On the Stability of a Discrete Time Ramsey Growth Model with Stochastic Labor

Gabriel Zacarías-Espinoza, Hugo Cruz-Suárez
and Enrique Lemus-Rodríguez *†‡

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Abstract

In this paper we study a version of Ramsey's discrete time Growth Model where the evolution of Labor through time is stochastic. Taking advantage of recent theoretical results in the field of Markov Decision Processes, a first set of conditions on the model are established that guarantee a long-term stable behaviour of the underlying Markov chain.

keywords: Discounted Markov Decision Processes, Stochastic Euler Equation, Stability of Markov Chains, ergodic convergence, Ramsey Growth Model.

1 Introduction

Ramsey's seminal work on economic growth has been extended in many ways, but, to the best of our knowledge, the study of a random discrete time version is still in its youth (see [18]) and provides an opportunity for a fruitful interaction between economists and mathematicians that should lead to better simulations and consequently, to a better understanding of the effects of the random deviations in the growth of an economy and its impact on the population. In this paper a first random model is proposed where the population, i.e., the Labor (force) grows in a stochastic manner.

The structure of the paper is simple: the model is spelled out, as well as a set underlying Assumptions, and its long-term stability is established. Due to space constraints, for the technical proofs the reader is referred to a theoretical paper ([20]) where a general framework is built.

*Gabriel Zacarías-Espinoza, Department of Mathematics, UAM-Iztapalapa, Ave. San Rafael Atlixco 186, Col. Vicentina, México 09340, México. e-mails: gabrielzaces@hotmail.com

†Hugo Cruz-Suárez, Facultad de Ciencias Físico-Matemáticas, Benemérita Universidad Autónoma de Puebla, Avenida San Claudio y Río Verde, Col. San Manuel, CU, 72570 Puebla, Pue., México. e-mail: hcs@fcfm.buap.mx

‡Enrique Lemus-Rodríguez, Escuela de Actuaría, Universidad Anáhuac México-Norte, Av. Universidad Anáhuac 46, Col. Lomas Anáhuac, 52786 Huixquilucan, Edo. de México. México. e-mail: elemus@anahuac.mx

2 Ramsey Growth Model driven by Stochastic Labor

Our model considers a random number of consumers, the population or Labor L_t, whose growth is Markovian:

$$\begin{aligned} L_{t+1} &= L_t \eta_t, \\ & L_0 \text{ is known} \end{aligned} \tag{1}$$

where L_t denotes the number of consumers at time t, $t = 0, 1, ...$, and $\{\eta_t\}$ is a sequence of independent and identically distributed (iid) random variables, such that $P(\eta_t > 0) = 1$.

Remark 2.1. *In the literature of economic growth models is usual to assume that the number of consumers grow very slowly in time (see [12] and [18]). Observe that the model presented in this paper is a first step in an effort to weaken that constraint of the model.*

The production function for the economy is given by

$$\begin{aligned} Y_t &= F(K_t, L_t), \\ & K_0 \text{ is known,} \end{aligned}$$

i.e. the production Y_t is a function of capital, K_t, and labor, L_t, where the production function, F, is a homogeneous function of degree one. The output must be split between consumptions $C_t = c_t L_t$ and the gross investment I_t, i.e.

$$C_t + I_t = Y_t. \tag{2}$$

Let $\delta \in (0, 1)$ be the depreciation rate of capital. Then the evolution equation for capital is given by:

$$K_{t+1} = (1 - \delta)K_t + I_t. \tag{3}$$

Substituting (3) in (2), it is obtained,

$$C_t - (1 - \delta)K_t + K_{t+1} = Y_t. \tag{4}$$

In the usual way, all variables can normalized into *per capita* terms, namely, $y_t := Y_t/L_t$ and $x_t := K_t/L_t$. Then (4) can be expressed in the following way:

$$c_t - (1 - \delta)x_t + K_{t+1}/L_t = y_t = F(x_t, 1).$$

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Now, using (1) in the previous relation, it is obtained

$$x_{t+1} = \xi_t(F(x_t, 1) + (1 - \delta)x_t - c_t),$$

$t = 0, 1, 2, ...,$ where $\xi_t := (\eta_t)^{-1}$. Define $h(x) := F(x, 1) + (1 - \delta)x$, $x \in X := [0, \infty)$, h henceforth to be identified as the production function.

The transition law of the system is given by

$$x_{t+1} = \xi_t(h(x_t) - c_t),$$

where $c_t \in [0, h(x_t)]$ and $\{\xi_t\}$ is a sequence of iid random variables, with $x_0 = x$ known.
Let ξ a generic element of $\{\xi_t\}$. Suppose that ξ has a density Δ, which is strictly positive and continuous second derivative, i.e, $\Delta \in C^2((0, \infty))$. Furthermore, suppose that $E[\xi^p]$ and $E[\xi^{-1}]$ exist and both are finite, where $p > 1$ and E denotes the expectation operator. Let A be defined by

$$A = \cup_{x \in X} [0, h(x)].$$

A is called control set.

Definition 2.2. *A plan or consumption sequence is a sequence $\pi = \{\pi_t\}_{t=0}^{\infty}$ of stochastic kernel π_t on the control set A given the history*

$$h_t = (x_1, c_1, ..., x_{t-1}, c_{t-1}, x_t), \tag{5}$$

for each $t = 0, 1, ...$ and satisfying the constraints $\pi_t(A(x_t)|h_t) = 1$, $t = 0, 1, \cdots$. The set of all plans will be denoted by Π.

Let $\mathbb{F}$ be the set of all measurable functions $f : X \to A$, such that $f(x) \in A(x)$ for every $x \in X$. A plan $\pi \in \Pi$ is *stationary* if there exists $f \in \mathbb{F}$ such that, under π, the control $f(x)$ is applied at each time $t = 0, 1, \ldots$. In this case, a stationary plan is denoted by f.

Given an initial capital $x = x_0 \in X$ and a plan $\pi \in \Pi$ then $\mathbb{P}_x^\pi$ denotes the probability measure on the canonical space $(\Omega, \Im)$, where $\Omega := (X \times A)^\infty$ and $\Im$ is their corresponding σ-algebra of Borel on Ω, where the performance index used to evaluate the quality of the plan π is determined by

$$v(\pi, x) = \mathbb{E}_x^\pi \left[\sum_{t=0}^{\infty} \alpha^t U(c_t) \right]$$

where $U : [0, \infty) \to \mathbb{R}$ is a measurable function known as utility function and $\alpha \in [0, 1]$ is the discount factor.

The goal of the controller is to maximize utility of consumption on all plans $\pi \in \Pi$, that is:

$$V(x) := \sup_{\pi \in \Pi} \mathbb{E}_x^\pi \left[\sum_{t=0}^{\infty} \alpha^t U(c_t) \right],$$

$x \in X$.

For ulterior reference, this model be called Ramsey Growth Model under Stochastic Labor, or **RSL**.

The RSL model is a Markov Decision Process (MDPs) (see [8] and [9]). In fact, the Markov Control Model could be identified in the following way: $X = A = [0, \infty)$, $A(x) = [0, h(x)]$, $x \in X$, the transition law Q is given by

$$Q(B|x, c) = \int I_B((h(x) - c)s) \Delta(s) ds,$$

$(x, c) \in \mathbb{K} := \{(x, c) | x \in X, c \in A(x)\}$. Futhermore, the reward function is given by an utility function U.
In this context, a plan is called a policy.

The following assumptions are well known in the context of economic growth models (see [2], [5], [11], [12] [13], [15], [16], [17], [19] and [18]), guaranteeing the existence of an optimal plan, the validity of the dynamic programming algorithm and a characterization of the optimal plan via a version of an stochastic Euler equation in the context of MDPs (see [5] and [6]). Through of this work, Assumptions 2.3 and 2.4 below, are supposed to hold.

Assumption 2.3. *The production function h, satisfies:*
$h \in C^2((0, \infty))$, is a concave function on X, $h' > 0$ and $h(0) = 0$ and,
Let $h'(0) := \lim_{x \downarrow 0} h'(x)$. Suppose that $h'(0) \geq 1$ and

$$\alpha h'(0) > E[\xi^{-1}]. \tag{6}$$

Assumption 2.4. *The utility function U satisfies:*

a) $U \in C^2((0, \infty), \mathbb{R})$, with $U' > 0$ and $U'' < 0$, $U'(0) = \infty$ and $U'(\infty) = 0$,

b) There is a function ϑ defined on S with $E[\vartheta(\xi)] < \infty$, and this satifies that

$$|U'(h(s(h(x) - c)))h'(s(h(x) - a))s\Delta(s)| \leq \vartheta(s), \tag{7}$$

$s \in S$, $c \in (0, h(x))$.

3 Basic Properties of the RSL

Assumptions 2.3 and 2.4 guarantee that two powerful tools, Dynamic Programming and Euler equation, can be used in the study of our RSL.

In paper [20], using results from [8], [9] and [10] it is shown that the standard Dynamic Programming techniques hold for the RSL, in particular, in [20] it is shown how the result in [3] ensure uniqueness of the optimal policy. Moreover, in [5] it is shown that for such optimal policy the optimal action is an interior point in $[0, h(x)]$ for each $x \in X$.

In short, under the above assumptions, the value function of the corresponding MDP satisfies Bellman's optimality equation, the optimal comsumption plan is unique and value iteration works. For details see [20].

Similarly, in [20], the results in [5] an [6] allow us to establish a MDP Euler equation in the context of the classical Value Iteration Algorithm, i.e.:

The optimal plan f satisfies

$$U'(f(x)) = \alpha E[h'(\xi(h(x) - f(x)))U'(f(\xi(h(x) - f(x))))\xi], \tag{8}$$

$x > 0$. Reciprocally, if $f \in \mathbb{F}$ satifies (8) and

$$\lim_{t\to\infty} \alpha^t E_x^f \left[h'(x_t)U'(f(x_t))x_t\right] = 0, \tag{9}$$

then f is an optimal plan.

For further results and details, see [20].

4 Stability of the RSL

Inspired by Nishimura and Stachurski in [16], it is possible to use Euler's equation ([20], [5] and [6]) to establish ergodic convergence towards a invariant probability measue of the optimal processse using density functions of the driving noise defined on $(0, \infty)$ the optimal process also converges in L^p if $p \geq 1$.

Let $f \in \mathbb{F}$ be the stationary optimal plan for the RSL model, the stochastic optimal process is given by

$$x_{t+1} = \xi_t(h(x_t) - f(x_t)),$$

$t = 0, 1, 2, \ldots, x_0 = x \in X = [0, \infty)$, known.

In order to avoid trivialities, we assume $x_0 > 0$. We know that:

$$Q(B|\, x, f(x)) = \int\limits_{\{s|s(h(x)-f(x))\in B\}} \Delta(s)ds, x > 0,$$

due to the fact that $h - f$ is strictly positive and that the density Δ is continuous and also strictly positive, we get that kernel $Q(\cdot|\, x, f(x))$ is determined by

$$q_1(y|\, x, f(x)) := \Delta\left(\frac{y}{h(x) - f(x)}\right)\frac{1}{h(x) - f(x)}, x > 0.$$

Given $x > 0$, inductively, for each time $t \geq 1$, x_t distribution has density q_t determined by

$$q_t(y) = \int q_1(y|\, x, f(x))q_{t-1}(x)dx.$$

For each $A \in B(X)$ we define measure:

$$\Xi(A) := \int\limits_B \Delta(s)ds.$$

Lemma 4.1. *The optimal process $\{x_n\}$ is Ξ-irreducible and strongly aperiodic.*

Proof. See [20] □

Let W be defined for$x \in (0, \infty)$ as

$$W(x) := [U'(f(x))h'(x)]^{1/2} + x^p + 1, \tag{10}$$

donde $p > 1$.

Lemma 4.2. *W is a Lyapunov function.*

Proof. See [20] □

From the previous results, in a dense technical deduction involving classical Markov Decision Processes techniques (see [20]) it is possible to prove the fundamental theorem of this paper, that due to space limitations, we will simply state:

Theorem 4.3. *There exists a unique invariant probability measure P for the optimal process of the RSL. In fact, this process is W-geometrically ergodic.*

Proof. See [20] □

And a very interesting side result is the following:

Theorem 4.4. *The corresponding optimal process of the RSL converges in mean.*

Proof. See [20]

□

5 Concluding remarks.

This first RSL has noteworthy stability properties: a unique invariant measure and geometric convergence. The Markov Decision Process Euler Equation approach is a powerful tool for the analysis of economic models like the RSL.

This paper is a first step in a much wider research programme where a further study of this and related models is pursued. In particular, actual simulations and actual calculation of the invariant probability measure should provide us with a better picture of the behaviour of this stochastic model. Also, the study of the robustness of the optimal policies, the trajectories and the invariant probability under perturbations of the model is of great importance.

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