Tangency-saddle singularities of Planar Bimodal Linear Systems

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***Abstract*—We continue the study of the bifurcations and the structural stability of planar bimodal linear dynamical systems (that is, systems consisting of two linear dynamics acting on each side of a straight line, assuming continuity along the separating line). Here we determine the tangency-saddle singularities in the saddle/spiral case, the only where they can appear.**

I. Introduction

Piecewise linear systems constitute a class of non-linear systems which have recently attracted the interest of researchers because of their interesting properties and the wide range of applications from which they arise. In [4] and [5] one studies the controllability of BLDS (bimodal linear dynamical systems). In [6] one begins the study of its structural stability. The structural stability of a system warrants that its qualitative behavior is preserved under small perturbations of their parameters. One focuses in planar BLDS, that is, two planar linear subsystems acting in complementary halfplanes, assuming continuity in the separating straight line. They have interesting theoretical properties as well as applications (see, for example, [1], [2], [4] and [7]).

In Section 3 we recall the results in [6]: by adapting the necessary conditions in [8], one obtains the list of possible structurally stable planar BLDS and one concludes that structural stability holds when (real) spirals do not appear; in addition, one studies the finite periodic orbits for the saddle/spiral case.

In Section 4 we enlarge this study to the tangency-saddle singularities. They can appear only in the saddle/spiral case. We prove that it does for a sequence of values of the trace of the spiral subsystem, whereas for the remainder values the BLDS is structurally stable.

Throughout the paper, $\mathbf{R}$ will denote the set of real numbers, $M_{n\times m}(\mathbf{R})$ the set of matrices having n rows and m columns and entries in $\mathbf{R}$ (in the case where $n = m$, we will simply write $M_n(\mathbf{R})$) and $Gl_n(\mathbf{R})$ the group of non-singular matrices in $M_n(\mathbf{R})$. Finally, we will denote by $e_1, \dots, e_n$ the natural basis of the Euclidean space $\mathbf{R}^n$.

II. Structurally stable planar bimodal linear systems

Let us consider a bimodal linear dynamical system given by

$$\{\ \dot{x}(t) = A_1x(t) + B_1 \qquad \text{if } Cx(t) \leq 0$$

$$\{\ \dot{x}(t) = A_2x(t) + B_2 \qquad \text{if } Cx(t) \geq 0$$

where $A_1, A_2 \in M_n(\mathbf{R})$; $B_1, B_2 \in M_{n\times 1}(\mathbf{R})$; $C \in M_{1\times n}(\mathbf{R})$. We assume that the dynamics is continuous along the separating hyperplane $H = \{x \in \mathbf{R}^n : Cx = 0\}$; that is to say, that both subsystems coincide for $Cx(t) = 0$.
By means of a linear change in the state variable $x(t)$, we can consider $C = (1\ 0 \dots 0) \in M_{1\times n}(\mathbf{R})$. Hence $H = \{x \in \mathbf{R}^n : x_1 = 0\}$ and continuity along H is equivalent to:

$$B_2 = B_1, \qquad A_2e_i = A_1e_i, \quad 2 \leq i \leq n.$$

We will write from now on $B = B_1 = B_2$.

Definition 1: In the above conditions, we say that the triple of matrices (A_1, A_2, B) defines a bimodal linear dynamical system. (BLDS.)

The placement of the equilibrium points will play a significative role in the dynamics of a BLDS. So, we define:

Definition 2: Let us assume that a subsystem of a BLDS has a unique equilibrium point, not lying in the separating hyperplane. We say that this equilibrium point is real *if it is located in the halfspace corresponding to the considered subsystem. Otherwise, we say that the equilibrium point is* virtual.

Our goal is to characterize the planar BLDS which are structurally stable in the sense of [8].

Definition 3: A triple of matrices (A_1, A_2, B) defining a BLDS is said to be (regularly) structurally stable if it has a neighborhood $V(A_1, A_2, B)$ such that for every $(A_1', A_2', B') \in V(A_1, A_2, B)$ there is a homeomorphism of $\mathbf{R}^2$ preserving the hyperplane H which maps the oriented orbits of (A_1', A_2', B') into those of (A_1, A_2, B) and it is differentiable when restricted to finite periodic orbits.

A natural tool in the study of BLDS is simplifying the matrices A_1, A_2, B by means of changes in the variables $x(t)$ which preserve the qualitative behavior of the system (in particular, the condition of structurally stability). So, we consider linear changes in the state variables space preserving the hyperplanes $x_1(t) = k$, which will be called *admissible basis changes*. Thus, they are basis changes given by a matrix $S \in Gl_n(\mathbf{R})$,

$$S = \begin{pmatrix} 1 & 0 \\ U & T \end{pmatrix}, \quad T \in Gl_{n-1}(\mathbf{R}), \quad U \in M_{n-1\times 1}(\mathbf{R}).$$

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See [3] for the resulting reduced forms.

Also, translations parallel to the hyperplane H are allowed.

III. Preliminaries

In [6] one proves the following results. Firstly,

Theorem 1: 1. The triples of matrices representing a structurally stable BLDS can be reduced (by means of an admissible basis change and a translation parallel to the separating line) to the form:

$$A_1 = \begin{pmatrix} T & 1 \\ -D & 0 \end{pmatrix}, A_2 = \begin{pmatrix} \tau & 1 \\ -\Delta & 0 \end{pmatrix}, B = \begin{pmatrix} 0 \\ b \end{pmatrix} \quad (*)$$

In particular, the only tangency point is $(0, 0)$.

2. The only possible structurally stable BLDS are those in Table 1.
3. A sufficient condition in order to be structurally stable is that none subsystem is a real spiral (cases 1, 2, 4, 5, 6, 8, 9, 10, 12, 13, 14 and 16).
4. In the case 3, it is structurally stable if and only if
   - 4.a the finite periodic orbits are hyperbolic and disjoint from the tangency points
   - 4.b there are not finite orbits connecting two saddles
   - 4.c there are not finite orbits connecting a saddle and a tangency point
5. In the cases 7, 11 and 15, it is structurally stable if and only if condition (a) holds.

Subsystem 1 \ 2	Virtual saddle	Real node	Real spiral	Real imp. node
Real saddle	1 ($b > 0$)	2 ($b > 0$)	3 ($b > 0$)	4 ($b > 0$)
Virtual node	5 ($b < 0$)	6 ($b > 0$)	7 ($b > 0$)	8 ($b > 0$)
Virtual spiral	9 ($b < 0$)	10 ($b < 0$)	11 ($b > 0$)	12 ($b > 0$)
Virtual imp. node	13 ($b < 0$)	14 ($b < 0$)	15 ($b < 0$)	16 ($b > 0$)

TABLE I.

Secondly, one focuses on conditions (a), (b) of case 3 for divergent spirals. Thus, let us assume a BLDS as in (*), verifying:

- The left subsystem is a (real) saddle, i.e.: $D < 0, b > 0$. In particular, its equilibrium point is $(\frac{b}{D}, -T\frac{b}{D})$, and the invariant manifold cut the separating line at $(0, -\frac{b}{\lambda_2})$ and $(0, -\frac{b}{\lambda_1})$, where $\lambda_2 < 0 < \lambda_1$ are the eigenvalues of A_1. ($\lambda_1 + \lambda_2 = T$, $\lambda_1\lambda_2 = D$.)
- The right subsystem is a (real) divergent spiral, i.e.: $\tau > 0, \tau^2 < 4\Delta, b > 0$. In particular, its equilibrium point is $(\frac{b}{\Delta}, -\tau\frac{b}{\Delta})$. We write $\alpha \pm i\beta$, $\beta > 0$ the eigenvalues of A_2. ($2\alpha = \tau$, $\alpha^2 + \beta^2 = \Delta$.)

Theorem 2: In the above conditions:

1.
   - 1.a If $T > 0$, then there is not homoclinic orbit.
   - 1.b If $T = 0$, then there is a homoclinic orbit only for $\tau = 0$, which is a not considered case.
   - 1.c If $T < 0$, the only homoclinic (i.e., saddle-loop) orbit appears for $\tau = \tau_H > 0$ verifying

$$\exp(\alpha t)\sin(\beta t - \varphi) + \frac{\beta}{M} = 0, \quad \pi + \varphi \le \beta t \le \frac{3\pi}{2} + \varphi$$

being

$$t = \frac{1}{\tau}\ln\left(\frac{\lambda_2^2}{\lambda_1^2}\frac{\lambda_1^2 - \tau\lambda_1 + \Delta}{\lambda_2^2 - \tau\lambda_2 + \Delta}\right)$$

where $M\cos(\varphi) = \alpha - \frac{\alpha^2+\beta^2}{\lambda_2}$, $M\sin(\varphi) = \beta$.
Moreover, $\tau_H > \frac{T\Delta}{D}$.

2.
   - 2.a If $T > 0$, then there are not finite periodic orbits.
   - 2.b If $T = 0$, then there are finite periodic orbits (all of them) only for $\tau = 0$, which is a not considered case.
   - 2.c If $T < 0$, a finite periodic orbit appears for $0 < \tau < \tau_H$, which is hyperbolic (indeed, attractive) and disjoint from the tangency points, and no saddle-tangency orbits appear.
3. In particular, the systems in case 3 with $T < 0$ and $0 < \tau < \tau_H$ are structurally stable.

IV. The tangency-saddle singularities

Here we tackle the case 3 for $T < 0$, $\tau > \tau_H$. We will see that there is a decreasing sequence $\tau_1, \tau_2, ... \to \tau_H$ of values of τ where tangency-saddle singularities appear. For the remainder values, the BLDS is structurally stable.

Theorem 3: In the conditions of case 3 and $T < 0$:

1. There exists a maximal value of τ, τ_1 (see Figure 1), for which a tangency-saddle orbit appears. This is for $\tau = \tau_1 > 0$ verifying

$$\exp(\alpha t)\sin(\beta t - \varphi) + \frac{\beta}{M} = 0, \quad \pi + \varphi \le \beta t \le \frac{3\pi}{2} + \varphi$$

being

$$t = \frac{1}{\tau}\ln\left(\frac{1 + \tau + \Delta}{\lambda_1^2}\right)$$

where $M\cos(\varphi) = \alpha$, $M\sin(\varphi) = \beta$.
Moreover, $\tau_1 > \lambda_1^2 - \Delta - 1$.
It is the only value of τ for which the tangent orbit at $(0, 0)$ has its first intersection with the separating hyperplane just at $(0, -b/\lambda_1)$.

2. There exists a decreasing sequence $(\tau_1, \tau_2, ..., \tau_k, ...) \to \tau_H$, $k \ge 1$ (see Figures 2 and 3), for which tangency-saddle orbits appear.
For the value $\tau = \tau_k$ the orbit starting in the tangency point $(0, 0)$ has its $(2k-1)$th intersection with the separating hyperplane just at $(0, -b/\lambda_1)$.
3. For the remainder values of $\tau > \tau_H$, the BLDS is structurally stable.

Proof:

1. From [6], the first intersection of a spiral passing through $(0, 0)$ with the hyperplane must verify

$$\exp(\alpha t)\sin(\beta t - \varphi) + \frac{\beta}{M} = 0, \quad \pi + \varphi \le \beta t \le \frac{3\pi}{2} + \varphi$$

where $M\cos(\varphi) = \alpha$, $M\sin(\varphi) = \beta$.
Moreover, again from [6], a spiral cuts $x_1 = 0$ in x_{21} and x_{22} if and only if

$$\exp(\mu t) = \frac{b + \mu x_{22}}{b + \mu x_{21}}$$

where $\mu = \alpha + i\beta$. Imposing that $x_{21} = 0$ and $x_{22} = -b/\lambda_1$ we get

$$t = \frac{1}{\tau} \ln(\frac{1 + \tau + \Delta}{\lambda_1^2})$$

and from it the bound for τ_1.

2. For $\tau = \tau_1$ the tangent orbit at $(0, 0)$ intersects $x_1 = 0$ just at $(0, -b/\lambda_1)$ (see Figure 1). When τ decreases, this (first) intersection point ascends, so that the orbit completes a full turn and intersects the axe $x_1 = 0$ twice between $(0, -b/\lambda_2)$ and $(0, -b/\lambda_1)$, and again under $(0, -b/\lambda_1)$ if $\tau - \tau_1$ is small enough. It is clear that this third intersection ascends when τ increases, so that for a certain (unique) value $\tau = \tau_2$ this third intersection point is just $(0, -b/\lambda_1)$ (see Figure 2).
Additional degrowth of τ gives a second turn (with two additional intersections between $(0, -b/\lambda_2)$ and $(0, -b/\lambda_1)$ and a fifth intersection with $x_1 = 0$ under $(0, -b/\lambda_1)$. As above, for a certain (unique) value $\tau = \tau_3$ this fifth intersection point is just $(0, -b/\lambda_1)$ (see Figure 3).
By recurrence, one obtains a sequence of decreasing values $\tau_1, \tau_2, ..., \tau_k, ...$ for which the tangent orbit at $(0, 0)$ intersects $x_1 = 0$ in $0, 1, ..., 2k - 2, ...$ points between $(0, -b/\lambda_2)$ and $(0, -b/\lambda_1)$, and another one just at $(0, -b/\lambda_1)$.
An analogous reasoning shows that $\lim \tau_k = \tau_H$: for $\tau = \tau_H$ the saddle orbit through $(0, -b/\lambda_2)$ intersects again $x_1 = 0$ at $(0, -b/\lambda_1)$, whereas the tangent orbit at $(0, 0)$ turns over the spiral toward this homoclinic orbit; for any slightly greater value $\tau_H + \epsilon$ the above saddle orbit intersects $x_1 = 0$ under $(0, -b/\lambda_1)$, so that the tangent orbit passes between this new intersection point and $(0, -b/\lambda_1)$; therefore, the reasoning in the above paragraph shows that there is some $\tau_k < \tau_H + \epsilon$.
3. By construction, for $\tau_k < \tau < \tau_{k+1}$ there are not tangency-saddle orbits. Moreover, the orbits for τ run between the ones for τ_k and τ_{k+1} so that neither finite periodic orbits nor saddle-tangency orbits can occur.

■

Exemp 1: For $T = -1, D = -1, \Delta = 5, b = 1$, we plot the tangency-saddle orbits: $\tau_1 = 1.145, \tau_2 = 0.782, \tau_3 = 0.745; \tau_H = 0.742$.

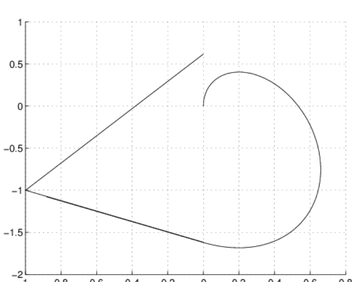

Fig. 1. Appearance of a tangency-saddle orbit: $T = -1, D = -1, \tau = \tau_1 = 1.145, \Delta = 5, b = 1$

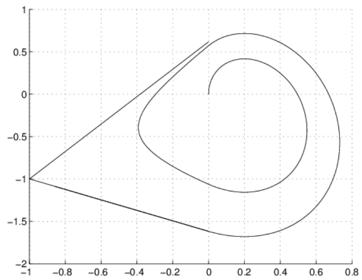

Fig. 2. Appearance of a tangency-saddle orbit: $T = -1, D = -1, \tau = \tau_2 = 0.782, \Delta = 5, b = 1$

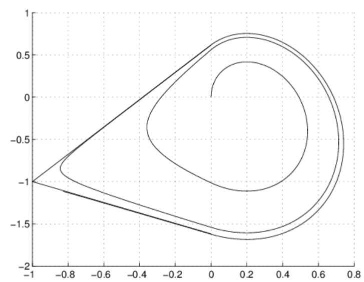

Fig. 3. Appearance of a tangency-saddle orbit: $T = -1, D = -1, \tau = \tau_3 = 0.745, \Delta = 5, b = 1$

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