Stability Analysis of Nonparallel Unsteady Flows by a Symmetry-Based Method

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Abstract: -
The problem of variables separation in the linear stability equations, which govern the disturbance behavior in viscous incompressible fluid flows, is treated using a symmetry-based approach. In the symmetry approach to the separations of variables in linear PDEs, a possibility of variable separation in a given PDE is intimately related to its symmetry properties. In the so-called direct approach to separation of variables in linear PDEs, which formalizes the main features of the symmetry approach, a form of the 'ansätz' for a solution with separated variables as well as a form of reduced ODEs, that should be obtained as a result of the variable separation, are postulated from the beginning. In the present study, such a direct approach is applied to the equations of the linear stability theory. The results of application of the method are the new coordinate systems and the most general forms of basic flows, which permit the postulated form of separation of variables. Thus, the stability analysis of nonparallel unsteady flows is reduced to the eigenvalue problems of ordinary differential equations. This method involves very complicated analytical calculations which can be implemented only using symbolic manipulating programs. The resulting eigenvalue problems are solved numerically with the help of the spectral collocation method based on Chebyshev polynomials. For some classes of perturbations, the eigenvalue problems can be solved analytically. Those unique examples of exact (explicit) solution of the nonparallel unsteady flow stability problems provide a very useful test for numerical methods of solution of eigenvalue problems, and for methods used in the hydrodynamic stability theory, in general.

Key-Words: - hydrodynamic stability, separation of variables, nonparallel flows, direct method, symbolic manipulating programs, spectral collocation method

1 Introduction

Problems of hydrodynamic stability are of great theoretical and practical interest, as evidenced by the number of publications devoted to this subject. The linear stability theory (see, e.g., [1]) for a particular flow starts with a solution of the equations of motion representing this basic flow. One then considers this solution with a small perturbation superimposed. Substituting the perturbed solution into the equations of motion and neglecting all terms that involve the square of the perturbation amplitude yield the linear stability equations which govern the behavior of the perturbation. The linearization provides a means of allowing for the many different forms that the disturbance can take. In the method of normal modes, small disturbances are resolved into modes, which may be treated separately because each satisfies the linear equations and there are no interactions between different modes.

Thus, the mathematical problem of the determination of stability of a given flow involves deriving a set of perturbation equations obtained from the Navier-Stokes equations by linearization around this basic flow and finding a set of possible solutions which would permit splitting a perturbation into normal modes. For a steady-state basic flow, normal modes depending on time exponentially, with a complex exponent λ, are permissable - the sign of the real part of λ indicates whether the disturbance grows or decays in time. If further separation of variables is possible, it makes the stability problem amenable to the normal mode analysis in its common form when the problem reduces to that of solving a set of ordinary differential equations. It can be done, however, only for basic flows of specific forms - mostly those are the parallel flows or their axial symmetric counterparts.

For nonparallel basic flows, when the coefficients in the equations for disturbance flow are dependent not only on the normal to the flow coordinate but also on the other coordinates, the corresponding operator does not separate unless certain terms are ignored. If, in addition, the

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basic flow is non-steady, this brings about great difficulties in theoretical studies of the instability since the normal modes containing an exponential time factor $exp(\lambda t)$ are not applicable here. Therefore stability of viscous incompressible flows developing both in space and time is a little studied topic in the theory of hydrodynamic stability.

All the above said shows that the method of separation of variables is of a fundamental importance for the hydrodynamic stability problems. Till now, the method of separation of variables has been used for stability analysis in an intuitive way which makes it generally applicable only to the stability problems of the steady-state parallel flows.

Recently, the so-called direct approach to separation of variables in linear PDEs has been developed by a proper formalizing the features of the notion of separation of variables (see, e.g., [2], [3]). This method involves very complicated analytical calculations which can be implemented only using symbolic manipulating programs. Till recently, computer capabilities were insufficient to apply this method to such complicated systems as linear stability equations – only single equations of mathematical physics have been treated. The first attempt of applying this method to the linear stability problem has been done in Ref. [4]. The success has been achieved not only due to the increase of computer capability but also at the expense of modifying the method based on physics of the problem. Solutions obtained has been used in Ref. [5] to implement the stability analysis of some viscous incompressible unsteady nonparallel flows, exact solutions of the continuity and Navier-Stokes equations in cylindrical coordinates.

In this paper, we present both some earlier results and the results obtained by further development of the stability analysis based on separation of variables in the linearized equations for the flow perturbations. Stability analysis of the three-dimensional unsteady nonparallel flows includes two stages. First, analytical calculations using the symbolic manipulating *Mathematica* package are made to determine classes of separable solutions for basic flows and separable solutions of the equations for perturbations – both in Cartesian and cylindrical coordinates. Next, the ODE eigenvalue problems to which the original stability problems reduce via separation of variables are solved numerically with the help of the spectral collocation method based on Chebyshev polynomials. The results obtained show dependence of the flow stability properties on the perturbation wave numbers and parameters of the problems. This includes neutral curves, perturbation spectra, unstable perturbation modes and others.

In some cases, the eigenvalue problems can be solved analytically. Those unique examples of exact (even explicit) solution of the nonparallel unsteady flow stability problems provide a very useful test for numerical methods of solution of eigenvalue problems, and for methods used in the hydrodynamic stability theory, in general.

2 Variable separation using the direct method

2.1 Formulation

The Navier-Stokes equations governing flows of incompressible Newtonian fluids are

$$\frac{\partial \hat{\mathbf{v}}}{\partial t} + (\hat{\mathbf{v}}\nabla)\,\hat{\mathbf{v}} = -\frac{1}{\rho}\nabla\hat{p} + \nu\nabla^2\hat{\mathbf{v}} \text{ and } \nabla\hat{\mathbf{v}} = 0, \quad (1)$$

where ρ is the constant density and ν is the constant kinematic viscosity of the fluid.

As usual in stability analysis, we split the velocity and pressure fields $(\hat{v}_x, \hat{v}_y, \hat{v}_z, \hat{p})$ into two problems: the basic flow problem (V_x, V_y, V_z, P) and a perturbation one (v_x, v_y, v_z, p),

$$\hat{v}_x = V_x + v_x,\ \hat{v}_y = V_y + v_y,\ \hat{v}_z = V_z + v_z,\ \hat{p} = P + p \quad (2)$$

Introducing (2) into the Navier-Stokes equations (1) and neglecting all terms that involve the square of the perturbation amplitude, while imposing the requirement that the basic flow variables (V_x, V_y, V_z, P) themselves satisfy the Navier-Stokes equations, one arrives at the following set of linear stability equations in the Cartesian coordinates:

$$\begin{gathered}
\frac{\partial v_x}{\partial t} + V_x\frac{\partial v_x}{\partial x} + v_x\frac{\partial V_x}{\partial x} + V_y\frac{\partial v_x}{\partial y} + v_y\frac{\partial V_x}{\partial y} + V_z\frac{\partial v_x}{\partial z} + \\
v_z\frac{\partial V_x}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right), \\
\frac{\partial v_y}{\partial t} + V_x\frac{\partial v_y}{\partial x} + v_x\frac{\partial V_y}{\partial x} + V_y\frac{\partial v_y}{\partial y} + v_y\frac{\partial V_y}{\partial y} + V_z\frac{\partial v_y}{\partial z} + \\
v_z\frac{\partial V_y}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right), \\
\frac{\partial v_z}{\partial t} + V_x\frac{\partial v_z}{\partial x} + v_x\frac{\partial V_z}{\partial x} + V_y\frac{\partial v_z}{\partial y} + v_y\frac{\partial V_z}{\partial y} + V_z\frac{\partial v_z}{\partial z} + \\
v_z\frac{\partial V_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right), \\
\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0, \quad (3)
\end{gathered}$$

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2.2 Outline of the direct method

Let us introduce a new coordinate system t, $\xi = \xi(t,x), \eta = \eta(t,y), \gamma = \gamma(t,z)$

We choose the Ansatz for a solution $\mathbf{u} = (v_x, v_y, v_z)$ and p to be found

$$\mathbf{u} = T(t)\exp(a\xi + s\gamma + mS(t))\mathbf{f}(\eta),$$
$$p = T_1(t)\exp(a\xi + s\gamma + mS(t))k(\eta) \qquad (4)$$

where $\mathbf{f} = (h, f, g)$ and functions $T(t)$, $T_1(t)$, $S(t)$, $\xi(t,x)$, $\eta(t,y)$, $\gamma(t,z)$ are not fixed *a priori* but chosen in such a way that inserting the expressions (4) into system of PDEs (3) yields a system of three second-order and one first order ordinary differential equations for four functions $h(\eta)$, $f(\eta)$, $g(\eta)$, $k(\eta)$. To get constraints on functions T, T_1, S, ξ, η, γ we formalize a reduction procedure as follows.

First, we postulate the form of the resulting system of ordinary differential equations as follows

$$\begin{aligned}
h''(\eta) &= U_{11}g'(\eta) + U_{12}h'(\eta) + U_{13}k'(\eta) + \\
&\quad U_{14}f(\eta) + U_{15}g(\eta) + U_{16}h(\eta) + U_{17}k(\eta),\\
f''(\eta) &= U_{21}g'(\eta) + U_{22}h'(\eta) + U_{23}k'(\eta) + \\
&\quad U_{24}f(\eta) + U_{25}g(\eta) + U_{26}h(\eta) + U_{27}k(\eta),\\
g''(\eta) &= U_{31}g'(\eta) + U_{32}h'(\eta) + U_{33}k'(\eta) + \qquad (5)\\
&\quad + U_{34}f(\eta)U_{35}g(\eta) + U_{36}h(\eta) + U_{37}k(\eta),\\
f'(\eta) &= U_{41}f(\eta) + U_{42}g(\eta) + U_{43}h(\eta) + U_{44}k(\eta).
\end{aligned}$$

Here U_{ij} are second order polynomials with respect to spectral parameters a, s, m with coefficients, which are some smooth functions on η and should be determined on the next steps of the algorithm. Next, we insert the expressions (4) into (3) which yields a system of PDEs containing the functions ξ, η, γ and their first- and second-order partial derivatives, and the functions $f(\eta)$, $g(\eta)$, $k(\eta)$ and their derivatives. Further we replace the derivatives $h''(\eta)$, $f''(\eta)$, $g''(\eta), f'(\eta)$ by the corresponding expressions from the right-hand sides of (5).

Now we regard $h'(\eta)$, $g'(\eta)$, $k'(\eta)$, $h(\eta)$, $f(\eta)$, $g(\eta)$, $k(\eta)$ as the new independent variables. As the functions $\xi(x,t)$, $\eta(y,t)$, $\gamma(z,t)$, $T(t)$, $T_1(t)$, $S(t)$, basic flows V_x, V_y, V_z and coefficients of the polynomials U_{ij} (which are functions of η) are independent on these variables, we can require that the obtained equality is transformed into identity under arbitrary $h'(\eta)$, $g'(\eta)$, $k'(\eta)$, $h(\eta)$, $f(\eta)$, $g(\eta)$, $k(\eta)$. In other words, we should split the equality with respect to these variables. After splitting we get an overdetermined system of nonlinear partial differential equations for unknown functions $\xi(x,t)$, $\eta(y,t)$, $\gamma(z,t)$, $T(t)$, $T_1(t)$, $S(t)$, basic flows V_x, V_y, V_z and coefficients of the polynomials U_{ij}. At the last step we solve the above system to get an exhaustive description of coordinate systems providing separability of equations (3), as well as all possible basic flows V_x, V_y, V_z such that the system (3) is solvable by the method of separation of variables.

Thus, the problem of variable separation in equation (3) reduces to integrating the overdetermined system of PDEs for unknown functions $\xi(x,t)$, $\eta(y,t)$, $\gamma(z,t)$, $T(t)$, $T_1(t)$, $S(t)$, basic flows V_x, V_y, V_z and coefficients of the polynomials U_{ij}. This have been done with the aid of *Mathematica* package.

2.3 Results for equations in Cartesian coordinates

The most general form of the basic flow is:

$$V_x = \nu A(\eta)T(t) - \frac{c_1'(t) + xT'(t)}{T(t)},$$
$$V_y = \nu B(\eta)T(t) - \frac{c_2'(t) + yT'(t)}{T(t)},$$
$$V_z = \nu C(\eta)T(t) - \frac{c_3'(t) + zT'(t)}{T(t)},$$

The forms of the perturbations v_x, v_y, v_z and p are:

$$\mathbf{u} = T(t)\exp\left(a\xi + s\gamma + m\int T(t)^2 dt\right)\mathbf{f}(\eta),$$
$$p = \rho\, T(t)^2 \exp\left(a\xi + s\gamma + m\int T(t)^2 dt\right)k(\eta),$$

where $\xi = T(t)x + c_1(t)$, $\eta = T(t)y + c_2(t)$, $\gamma = T(t)z + c_3(t)$.

The equations with separated variables are

$$\begin{aligned}
&(m - a^2\nu - s^2\nu + a\nu A(\eta) + s\nu C(\eta))h(\eta) + \\
&\quad ak(\eta) + \nu(f(\eta)A'(\eta) + B(\eta)h'(\eta) - h''(\eta)) = 0,\\
&f(\eta)(m - a^2\nu - s^2\nu + a\nu A(\eta) + s\nu C(\eta) + \\
&\quad \nu B'(\eta)) + \nu B(\eta)f'(\eta) + k'(\eta) - \nu f''(\eta) = 0,\\
&(m - a^2\nu - s^2\nu + a\nu A(\eta) + s\nu C(\eta))g(\eta) + \\
&\quad sk(\eta) + \nu(f(\eta)C'(\eta) + B(\eta)g'(\eta) - g''(\eta)) = 0,\\
&sg(\eta) + ah(\eta) + f'(\eta) = 0. \qquad (6)
\end{aligned}$$

The restrictions on the forms of the basic flows following from the requirement that they themselves satisfy the Navier-Stokes equations lead to the two following cases:

Case I:

$$\xi = \frac{1}{\sqrt{t}}x + c_1(t);\ \eta = \frac{1}{\sqrt{t}}y + c_2(t);\ \gamma = \frac{1}{\sqrt{t}}z + c_3(t).$$

$$\begin{aligned} V_x &= \frac{x}{2t} + \nu A(\eta)\frac{1}{\sqrt{t}} - c_1'(t)\sqrt{t}, \\ V_y &= -\frac{y}{t} - \frac{1}{\sqrt{t}}\left(tc_2'(t) + \frac{3}{2}c_2(t)\right), \\ V_z &= \frac{z}{2t} + \nu C(\eta)\frac{1}{\sqrt{t}} - c_3'(t)\sqrt{t}, \end{aligned} \quad (7)$$

and the functions $A(x)$ and $C(x)$ satisfy the equations

$$3A'(\eta) + 3\eta A''(\eta) + 2\nu A'''(\eta) = 0, \quad (8)$$

$$3C'(\eta) + 3\eta C''(\eta) + 2\nu C'''(\eta) = 0. \quad (9)$$

which can be solved in terms of the error functions and the generalized hypergeometric functions. The separation Ansatz takes the form

$$\mathbf{u} = t^s e^{a\xi + m\gamma}\mathbf{f}(\eta),\ p = \rho t^{s-1/2}e^{a\xi+m\gamma}\pi(\eta). \quad (10)$$

For the *Case II* we have $\xi = x + c_1(t);\ \eta = y + c_2(t);\ \gamma = z + c_3(t);\ V_x = A_1\eta^2 + A_2\eta - c_1'(t)$, $V_y = -c_2'(t)$, $V_z = C_1\eta^2 + C_2\eta - c_3'(t)$ and the separation Ansatz is

$$\mathbf{u} = e^{a\xi + s\gamma + mt}\mathbf{f}(\eta),\ p = \rho e^{a\xi+s\gamma+mt}\pi(\eta).$$

2.4 Results for equations in cylindrical coordinates

The Navier-Stokes equations are written in cylindrical coordinates (r, φ, z) and then the velocity and pressure fields $\hat{v}_r, \hat{v}_\varphi, \hat{v}_z, \hat{p}$ are splitted into the basic flow and perturbation parts

$$\hat{v}_r = V_r + v_r,\ \hat{v}_\varphi = V_\varphi + v_\varphi,\ \hat{v}_z = V_z + v_z,\ \hat{p} = P + p \quad (11)$$

where V_r, V_φ, V_z, P are the basic flow fields and v_r, v_φ, v_z, p are the perturbations.

Application of the direct method defines the forms of the basic flows allowing separation of variables in the stability equations as well as the solutions of the stability equations with separated variables.

The most general form of the basic flow is:

$$\begin{aligned} V_z &= A(\xi)T(t) - \frac{c'(t) + zT'(t)}{T(t)}, \\ V_r &= B(\xi)T(t) - r\frac{T'(t)}{T(t)},\ V_\varphi = C(\xi)T(t), \end{aligned} \quad (12)$$

where $\xi = T(t)r$, $\eta = T(t)z + c(t)$.

The forms of the perturbations $\mathbf{u} = (v_r, v_\varphi, v_z)$ and p and with the trial functions $\mathbf{f} = (f, g, h)$ and π we have:

$$\begin{aligned} \mathbf{u} &= T(t)\exp\left(a\eta + m\varphi + s\int T(t)^2 dt\right)\mathbf{f}(\xi), \\ p &= \rho T(t)^2 \exp\left(a\eta + m\varphi + s\int T(t)^2 dt\right)\pi(\xi). \end{aligned} \quad (13)$$

The restrictions on the forms of the basic flows following from the requirement that they satisfy Navier-Stokes equations lead to the two cases similar to those obtained for equations in Cartesian coordinates.

3 Stability properties of some flows

3.1 Basic flow

Considering the class of solutions in Cartesian coordinates identified in Section 2 as Case I, we will specify the solutions by setting $c_1(t) = c_2(t) = c_3(t) = 0$ but will use a possibility to enrich the solutions by a shift of the time variable. Making change of variables $t = t' - 1/b$, where b is a constant, and omitting primes in what follows, we will have the solution of the Navier-Stokes equations in Cartesian coordinates in the form

$$\begin{aligned} V_x &= \frac{1}{\sqrt{1-bt}}\left(-\frac{b\xi}{2} + \nu A(\eta)\right), V_y = \frac{b\eta}{\sqrt{1-bt}}, \\ V_z &= \frac{1}{\sqrt{1-bt}}\left(-\frac{b\zeta}{2} + \nu C(\eta)\right), \\ \frac{P}{\rho} &= \frac{1}{1-bt}\times \\ &\times\left(\frac{b^2}{8}\left(\xi^2 + \zeta^2 - 8\eta^2\right) - 2\nu^2\left(A_3\xi + C_3\zeta\right)\right), \end{aligned}$$

where

$$\xi = \frac{x}{\sqrt{1-bt}}, \quad \eta = \frac{y}{\sqrt{1-bt}}, \quad \zeta = \frac{z}{\sqrt{1-bt}} \quad (14)$$

and b can be both positive and negative. The functions $A(\eta)$ and $C(\eta)$ are given in terms of the error $\mathrm{erf}(z)$ and generalized hypergeometric ${}_2F_2(z)$ functions. The perturbations (10) must be correspondingly specified.

The above formulas remain valid if we introduce the nondimensional variables, with the time scale $1/|b|$ and the correspondingly defined velocity scale. In the dimensionless equations (we will retain the same notation for the nondimensional variables), the parameter b takes one of the two values: $b = 1$ or $b = -1$, and ν is replaced by

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1/Re where Re is the Reynolds number. (if we mark the dimensional variables with stars, the Reynolds number will be $\mathrm{Re} = L^{*2}|b^*|/\nu^*$ where L^* is the length scale.)

We will consider the solution for the case of $b = -1$ which allows interpretations corresponding to unsteady flows near stretching (impermeable or permeable) surfaces or the flows that develop within a channel possessing permeable, moving walls. It is worth remarking that the considered flows are essentially nonparallel – the flow fields include all three velocity components dependent on all coordinates.

There exists a class of solutions of the Navier-Stokes equations in cylindrical coordinates, which is similar in many features to the class of solutions in Cartesian coordinates considered above. The basic flow solution in cylindrical coordinates permits interpretations similar to those considered above for the solution in Cartesian coordinates. However, the cylindrical geometry and presence of the additional free parameters allow one to find more problem formulations and enrich the problem definitions. The basic flow might be again an unsteady axially symmetrical stagnation-point type flow, with the flow velocity decreasing with time as $(1 + t)^{-1}$, but, as distinct from the flows considered in the previous section, here fluid flows radially from infinity approaching the axis and spreading along it. The basic flow might also be an unsteady flow inside an expanding stretching cylinder, which may also rotate, and there is an injection of fluid through the porous pipe surface.

3.2 Criterion for stability

We choose as a criterion for stability that the ratio of the magnitude of a perturbation to that of a basic flow decreases with time, which for the solutions leads to

$$\Re\left(s + \frac{1}{2}\right) < 0 \quad \text{or} \quad \Re(s) < -\frac{1}{2} \tag{15}$$

where $\Re(s)$ denotes a real part of the eigenvalue s (the imaginary part $\Im(s)$, if nonzero, determines the oscillation frequency). In particular, for the decelerating flow ($b = -1$) the meaning of instability implies that even any disturbance is damped ($\Re(s) < 0$ for the velocity perturbations and $\Re(s) < 1/2$ for the pressure perturbations) yet it may dominate the decelerating flow after sufficient time if $\Re(s) > -1/2$. It is also seen that the condition (15) unifies the stability criterion for the velocity and pressure perturbations.

3.3 Solution of the eigenvalue problems

The eigenvalue problems were solved numerically with the help of the spectral collocation method based on Chebyshev polynomials [6, 7]. For some classes of perturbations, the eigenvalue problems can be solved analytically (see below) which provides an additional, probably the most important, testing the numerical results.

It can be shown that there exists a transformation (similar in a sense to Squire's transformation [1]) such that the three-dimensional problem defined by equations (6) can be reduced to an equivalent two-dimensional problem. Then equations for the perturbation amplitudes can be reduced to a system of two equations for two functions $g(\eta)$ and $h(\eta)$ of the form

$$\begin{aligned}&\alpha\left(\alpha b - \alpha b s + \alpha^3\nu + i\nu\left(\alpha^2 A(\eta) + A''(\eta)\right)\right) g(\eta) + \\ &\frac{3}{2} b\alpha^2 \eta g'(\eta) - \left(b - bs + 2\alpha^2\nu + i\alpha\nu A(\eta)\right) g''(\eta) - \\ &\quad \frac{3}{2} b\eta g'''(\eta) + \nu g^{(\mathrm{IV})}(\eta) = 0,\end{aligned} \tag{16}$$

$$\begin{aligned}&\nu C'(\eta) g(\eta) + \left(-\frac{1}{2} b - bs + \alpha^2\nu + i\alpha\nu A(\eta)\right) h(\eta) + \\ &\quad \frac{3}{2} b\eta h'(\eta) - \nu h''(\eta) = 0\end{aligned} \tag{17}$$

It is seen that for $C(\eta) = 0$ the system of equations (16) and (17) decouples into two separate equations for $g(\eta)$ and $h(\eta)$. Thus, in this case two separate branches exist, first of which corresponds to the disturbances with one z-component of the velocity vector changing with x and y, while the second branch corresponds to the two-dimensional disturbances with velocity vector lying in the (x, y) plane and not dependent on z.

In the case where both $A(\eta) = 0$ and $C(\eta) = 0$ equations (16) and (17) can be reduced to Kummer's equation [8] and can be solved in quadratures in terms of confluent hypergeometric functions.

There is an important point in which the stability problems in cylindrical coordinates differ from those in Cartesian coordinates: a transformation, similar to Squire's transformation, which reduces the three-dimensional perturbation problem to an equivalent two-dimensional problem, does not exist. Therefore, in general, one has to consider the three-dimensional perturbations to assess the flow stability. Below we present the results of numerical solution of the eigenvalue problems for the most general three-dimensional perturbations of the unsteady nonparallel flows developing within expanding pipe.

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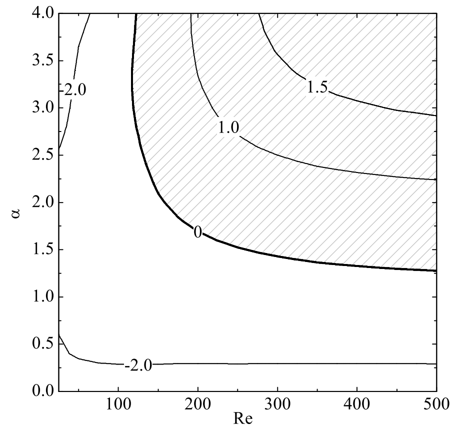

Figure 1: Neutral curve and contours of constant growth rate S for $U_0 = 30$ and $n = 2$. The shaded area represents the region in parameter space where unstable solutions exist.

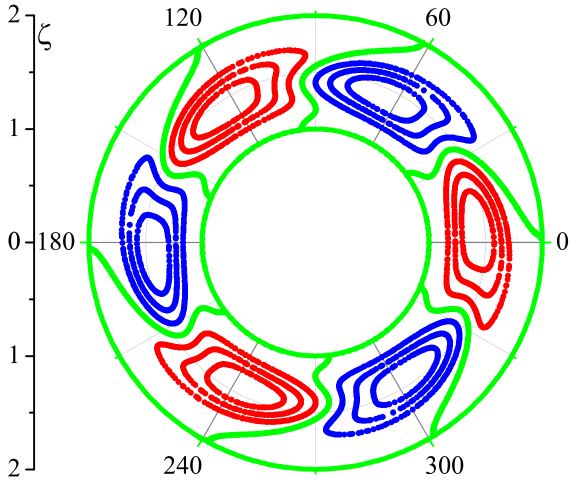

Figure 2: Flow in a gap. The unstable mode. Red, blue and green lines correspond to negative, positive and zero values of stream function.

First, the analysis shows that the flow within not rotating cylinder and in the absence of the axial pressure gradient is stable ($S < 0$) in all the parameter space. All the eigenvalues are real so that the disturbances decay monotonically.

If the basic flow includes the part due to the axial pressure gradient ($U_0 \neq 0$), positive values of S appear (see Fig. 1). The neutral curve $S = 0$ in Fig. 1 separates the regions of stability and instability. It is seen that for any Reynolds number larger than some critical value Re_* (for $U_0 = 30$, $\mathrm{Re}_* \approx 120$) there exists a range of wave numbers α corresponding to unstable solutions. Thus, the flow including the part due to the axial pressure gradient is unstable for $\mathrm{Re} > \mathrm{Re}_*$. The critical Reynolds number Re_* decreases while U_0 increases. Another example of application of the method is the flow n a gap between rotating expanding cylinders. An unstable mode for this flow is given in Fig. 2.

4 Concluding remarks

To conclude, in this paper we present a unified, computational synthesis of analytical and numerical calculations to study stability of viscous unsteady nonparallel flows. The combination of analytical and numerical solutions may provide a basis for a well-grounded discussion of some problematic points of hydrodynamic stability analysis and a very useful test for methods used in the hydrodynamic stability theory, in general. It is also worth remarking that the basic flows whose stability is studied in the paper are themselves of interest for fluid dynamics and have received considerable attention in the literature due to their relevance in a number of engineering applications.

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