Second-order sliding mode control applied to an inverted pendulum

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Abstract—In this paper first and second order sliding mode controllers for underactuated manipulators are proposed. Sliding mode control SMC is considered as an effective tool in different studies for control systems. However, the associated chattering phenomenon degrades the system performances. To overcome this phenomenon and to track a desired trajectory a twisting and a super-twisting algorithms are presented. The stability analysis is done using a Lyapunov function for the proposed controllers. Further, 3 different controllers are compared. As an illustration, the example of an inverted pendulum is considered. Simulation results are given to demonstrate the effectiveness of the proposed approaches.

Keywords—Underactuated manipulator, sliding mode control, twisting algorithm, super-twisting algorithm, inverted pendulum.

I. INTRODUCTION

UNDERACTUATED Mechanical Systems UMS are increasingly present in the robotic field. They have less actuators than degrees of freedom. In these systems, we find manipulators, vehicles and humanoids with several passive joints. Underactuations arise by deliberating in the design for the purpose of reducing the weight of the manipulator or might be caused by actuator failures. The difficulty in controlling underactuated mechanisms is due to the fact that techniques, developed for fully actuated systems, cannot be directly used. These systems are not feedback linearizable, yet they exhibit nonholonomic constraints and nonminimum phase characteristics [1]. Moreover, it has been shown that it is difficult to stabilize this class of systems by classical continuous controllers. This yields to the fact that the class of underactuated mechanical systems present challenging control problems. One of the common methods used to control underactuated systems is the sliding mode control SMC based on Lyapunov design. The SMC has been always considered as an efficient approach in control systems, due to its high accuracy and robustness with respect to various internal and external disturbances. The SMC approach consists in two steps. The first one is to choose a manifold in the state space that forces the state trajectories to remain along it. The second one is to design a discontinuous state-feedback capable of forcing the system to reach the state on the manifold in finite time. However, the drawback of the SMC is the presence of the chattering effect, caused by the switching frequency of the control [2]. The high frequency components of the control propagate on the system, to excite the unmodeled fast dynamics and therefore to cause undesired oscillations. In fact, this can degrade the system performances or may even lead to instability. In the literature, three main approaches has been presented, that help to reduce the chattering effects. The class of methods consists in the use of the saturation control instead of the discontinuous one. It ensures the convergence to a boundary layer of the sliding manifold. Moreover, in [3] and [4], a switching function, inside the boundary layer of the sliding manifold, has been approximated by a linear feedback gain. However, the accuracy and the robustness of the sliding mode are partially lost.

The second class of methods consist in the use of a system observer-based approach [5]. It can reduce the problem of robust control to the problem of exact robust estimation. This phenomenon can lead to the deterioration of the robustness with respect to the plant uncertainties or disturbances. Using the high order sliding mode controllers given by Levant [6], [7] as a way to reduce the chartering phenomenon and to keep the main advantages of the original approach of the SMC is another way to eliminate chattering. The high order sliding mode consists in the sliding variable system derivations. It maintains the robustness of the system. Specially, the second order sliding mode control is relatively simple to implement and it gives good robustness to external disturbances. The second-order sliding-mode SSMC approach can reduce the number of differentiator stages in the controller. However, the stability proofs are based usually on a geometrical or homogeneity methods since the Lyapunov function is a difficult task to define [8]. The stability and the convergence using SSMC is a challenge and several trials were made to deal with those difficulties. Recently, in [9] authors construct a Lyapunov function. It provides a finite time convergence, a robustness and an estimate of the convergence time for super twisting algorithm. In [10], a multivariable super twisting structure is proposed, which analyses the stability using the ideas of Lyapunov function given in [9].

Inverted pendulum system is a typical benchmark of non-linear underactuated mechanical systems [11]. For this system, the control input is the force $u$ that moves the cart horizontally and the output is the angular position of the pendulum $\theta$. Therefore, the inverted pendulum has been a popular candidate to illustrate different control methods. However, despite its simple mechanical structure, this prototype is not easy to control and it requires sufficiently sophisticated control designs. Indeed it is proven that the system is not feedback linearizable and has no corresponding constant relative degree [12]. Moreover, Zhao and Spong [13] have shown that several geometric properties of the system are lost when the pendulum...
moves through horizontal positions. The principal control task considered for the cart pendulum is to swing up the pendulum from the stable equilibrium point to the unstable equilibrium point, and stabilizing the cart in a desired position. In general, the main difficulty is to swing up the pendulum from the downward vertical position and to keep the cart stable. Numerous control techniques have been evolved to stabilize the inverted pendulum such as Proportional-Integral-Derivative (PID) controllers where the control gains are adjustable and updated online with a stable adaptation mechanism [14].

In this paper, the objective is to develop a robust position tracking controller based on the first and the second order sliding mode approach applied to an inverted pendulum. Stability of the closed loop system is carried out using a candidate Lyapunov function for the proposed controllers. The paper is organized as follows. Section 2 describes the model of the inverted pendulum and the first sliding mode controller. Section 3 deals with the sliding mode controllers and the design of second order sliding mode controllers. Section 4 discusses the simulation results of the proposed controllers.

II. DYNAMIC MODEL AND CONTROL APPROACH FOR AN INVERTED PENDULUM

A. Dynamic model

The dynamical behavior of inverted pendulum can be described by the following differential equations [13]:

\[(m + M)\ddot{y} + ml(\dot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta) = \tau \]
\[\ddot{y} \cos \theta + l\ddot{\theta} + g \sin \theta = 0 \quad (1)\]

where \(l\) is the length of the pendulum, \(m\) is the pendulum mass, \(M\) is the cart mass, \(\tau\) is the horizontal force action, \(\theta\) is the angular deviation, \(y\) is the position of the cart which is moving horizontally.

Fig. 1. Inverted pendulum

Letting \(x_1 = y\), \(x_2 = \dot{y}\), \(x_3 = \theta\), \(x_4 = \dot{\theta}\), and according to the canonical form of a class of underactuated systems, we can transform equations (1) into the following state space representation:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= f_1 + b_1 \tau \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= f_2 + b_2 \tau \\
\end{align*}
\]

\(x = [x_1, x_2, x_3, x_4]^T\) is the state variable vector, \(\tau\) is the control input. \(f_1, f_2, b_1\) and \(b_2\) are nominal nonlinear functions, described as:

\[
\begin{align*}
f_1 &= \frac{ml^2 x_3 - mg \sin x_3 \cos x_3}{M + ml^2} \\
f_2 &= \frac{(m + M)g \sin x_3 - ml^2 \cos x_3}{l(M + ml^2)} \\
b_1 &= \frac{1}{M + ml^2}; \quad b_2 = \frac{-\cos x_3}{l(M + ml^2)}
\end{align*}
\]

(3)

Letting
\[\tau = M + ml^2 x_3 u - (ml^2 x_3 - mg \sin x_3 \cos x_3)\]
equation (3) becomes:

\[
\dot{X} = f(x) + g(x)u
\]

(5)

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4 \\
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & g \sin x_3 \\
0 & 0 & \frac{gLx_3}{l} & 0
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix} + \begin{bmatrix}
0 \\
1 \\
0 \\
-\cos x_3
\end{bmatrix} u
\]

(6)

[15], [16] proposed a method that can approximate the original system with an input-output linearizable control system in new coordinates. This stabilization method of nonlinear system using sliding mode control, is based on coordinate transformation by the mapping \(T : x \mapsto \xi\) defined by:

\[
\xi_i = L_f^{-1} h(x), i \in 1, 2, 3, 4
\]

(7)

with \(\xi = (\xi_1, \xi_2, \xi_3, \xi_4)^T\). \(T\) is defined as a local diffeomorphism with \(T(0) = 0\).

The dynamical system in the new coordinates can be approximated by the system model:

\[
\begin{align*}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
\dot{\xi}_3 &= \xi_4 \\
\dot{\xi}_4 &= L_f h(x) + L_f h(T^{-1}(\xi))u
\end{align*}
\]

(8)

\(L_f h(x)\) is the Lie derivative of \(h(x)\) along the vector \(f(x)\). Consider the output system function of (5) defined by [16]:

\[
z = h(x) = x_1 + l \ln \left(\frac{1 + \sin x_3}{\cos x_3}\right)
\]

(9)
with \( \xi = T(x) \) and \( T_1(x) = h(x) = \xi_1 \). Define the transformation \( T : x \mapsto \xi \) by:

\[
T(x) = \begin{bmatrix} h(x) \\ L_1 h(x) \\ L_2^2 h(x) \\ L_3^3 h(x) \end{bmatrix} = \begin{bmatrix} \xi_1 = T_1(x) \\ \xi_2 = T_2(x) \\ \xi_3 = T_3(x) \\ \xi_4 = T_4(x) \end{bmatrix} = \xi
\]

then

\[
T(x) = \begin{bmatrix} x_1 + \ln \left( \frac{1 + \sin x_3}{\cos x_3} \right) + \frac{lx_4}{\cos x_3} \\ \tan x_3 \left( g + \frac{lx_4}{\cos x_3} \right) \\ \frac{2}{\cos^2 x_3} - \frac{1}{\cos x_3} \right) + \left( \frac{3g}{\cos^2 x_3} - 2g \right) x_4 - 2x_4 \tan x_3 u \\ \right)
\]

\[
\dot{\xi}_4 = z^4 = f_c(x) + g_c(x) u
\]

where

\[
f_c(x) = \left( \frac{6 \sin x_3}{\cos^4 x_3} - \frac{\sin x_3}{\cos^2 x_3} \right) t^4 x_4 + \frac{6g \sin x_3}{\cos^3 x_3} x_4^2 \\
+ \left( \frac{2g \sin x_3}{\cos^3 x_3} - \frac{g \sin x_3}{\cos x_3} \right) 3x_4^2 \\
+ \left( \frac{3g}{\cos^2 x_3} - 2g \right) \frac{g \sin x_3}{l} \\
g_c(x) = \frac{6x_4}{\cos^2 x_3} + 3x_4^2 - \frac{3g}{l \cos x_3} x_3^2 + \frac{2g \cos x_3}{l}
\]

by neglecting \( 2x_4 \tan(x_3) \) [16], we obtain a feedback linearizable nonlinear system in the state \( \xi \), with:

\[
\dot{\xi}_1 = \xi_2 \\
\dot{\xi}_2 = \xi_3 \\
\dot{\xi}_3 = \xi_4 \\
\dot{\xi}_4 = f_c(x) + g_c(x) u \\
z = \xi_1
\]

B. First order sliding mode controller

Define the surface \( s = \{ \xi \in R^4 | s(\xi) = 0 \} \), for \( \lambda > 0 \).

\[
s(\xi) = \left( \frac{d}{dt} + \lambda \right)^3 (z - z_d)
\]

We choose \( z_d = [2 \ 0 \ 0 \ 0]^T \). The time derivative of \( s \) along the system trajectory \( \dot{\xi} \) is equal to:

\[
\dot{s}(\xi) = \xi(4) + 3\lambda \xi(3) + 3\lambda^2 \xi(2) + \lambda^3 \xi(1) = f_c(\xi) + g_c(\xi) u + 3\lambda z(3) + 3\lambda^2 z(2) + \lambda^3 z(1)
\]

The sliding mode control is expressed by :

\[
u = u_{eq} + u_{sw}
\]

where \( u_{sw} \) is the switching control and \( u_{eq} \) is the equivalent control yielded from \( \dot{s}(\xi) = 0 \):

\[
u_{eq} = -f_c(z) + 3\lambda z(3) + 3\lambda^2 z(2) + \lambda^3 z
\]

\[
u_{sw} = \eta \text{sign}(s) + ks
\]

where \( \eta \) and \( K \) are positive constants.

It is notable that for small deviations, we have : \( g_c(\xi) < -3 - \frac{z}{\lambda} < 0 \). Choosing the Lyapunov candidate as:

\[
V = \frac{1}{2} s^2
\]

Differentiating \( V \) along the trajectories of (14) yields to

\[
\dot{V} = ss = -\eta |s| - ks^2 \leq 0
\]

Then the system is stable and the convergence of the sliding mode is guaranteed.

III. Second order sliding mode controller

The drawback of the first order sliding mode control is the chattering phenomenon. As a solution to resolve this problem, a higher order sliding mode HOS is proposed. In fact, HOS appears as an effective application to counteract the chattering phenomenon and the switching control signals, with higher relative degrees in finite time [8], [18].

The HOS has been introduced by Emel’yanov et al. [6], with the goal to get a finite time on the sliding set of order \( r \) defined by:

\[
s = \dot{s} = \ddot{s} = \ldots = s^{(r-1)} = 0.
\]

\( s \) defines the sliding variable with the \( r \)th order sliding and with its \( (r-1) \) first time derivatives depending only on the state \( x \). The first order sliding mode tries to keep \( s = 0 \). In the case of second order sliding mode control SSMC, which only needs its measurement or evaluation of \( s \), the following relation should be verified:

\[
s(x) = \dot{s}(x) = 0
\]

In the following, twisting algorithm and the super-twisting algorithm with a prescribed convergence law are used.
A. Twisting controller

1) Controller Approach: Consider the sliding surface

\[ s_1 = \left( \frac{d}{dt} + \lambda_1 \right)^2 \xi \]  

(22)

Differentiating twice (22) gives:

\[ \ddot{s}_1 = f_e(\xi) + g_e(\xi)u + 2\lambda_1 \dot{z}(3) + \lambda_1^2 \dot{z}(2) \]
\[ \ddot{s}_1 = \Psi(\xi) + \varphi u \]  

(23)

where \( \Psi(\xi) = f_e(\xi) + 2\lambda_1 \dot{z}(3) + \lambda_1 \dot{z}(2) \cdot \varphi(\xi) = g_e(\xi) \).

we assume that function \( \Psi \) and \( \varphi \) are bounded as:

\[ 0 < \varphi_m \leq \varphi \leq \varphi_M, \]

\[ \Psi_d, \varphi_m, \varphi_M \] and \( \varphi_d \) are positive scalars. Then, we have

\[ \frac{|\Psi|}{\varphi} < \frac{\Psi_d}{\varphi_M} \]  

(25)

By annulling \( \ddot{s}_1 = 0 \) the equivalent control can be expressed as: \( u_{eq} = -\Psi/\varphi \).

2) Stability study: The dynamic control law using the twisting algorithm is given by [8]:

\[ u_{sw} = \frac{K}{g_e(\xi)}(s_1 + \beta \text{sign}(\dot{s}_1)) \]  

(26)

with \( \beta > 0, 0 < K \leq K_M \) and \( K_M > \frac{1}{1-\beta \varphi_M} \). The total control is defined by:

\[ u = u_{eq} + u_{sw} \]  

(27)

The Lyapunov function can be chosen as:

\[ V_1 = \frac{1}{2} \lambda_2 s_1^2 + \frac{1}{2} \dot{s}_1^2 \]  

(28)

Differentiating (28) yields:

\[ \dot{V}_1 = \lambda_2 \dot{s}_1 s_1 + \dot{s}_1 \dot{s}_1 \]
\[ = \lambda_2 s_1^2 + \dot{s}_1(\Psi(\xi) + g_e(\xi)u) \]
\[ = \dot{s}_1[\lambda_2 s_1 - K s_1 - K \beta \text{sign}(s_1)] \]
\[ = s_1 \text{sign}(s_1)[\lambda_2 s_1 \text{sign}(s_1) - K s_1 \text{sign}(s_1) - K \beta] \]
\[ = |s_1|[\lambda_2 |s_1| - K |s_1| \text{sign}(s_1) - K \beta] | \]
\[ \leq |s_1|[\lambda_2 - K] |s_1| - K \beta] \]
\[ \leq 0 \]  

(29)

Therefore the system is stable if \( \lambda_2 - K < 0 \).

B. Super twisting controller

SSMC controllers require the knowledge of values of the derivatives except for the super twisting algorithm (STW). The STW is a continuous SM algorithm ensuring main properties of the first order sliding mode control for systems with Lipschitz continuous matched uncertainties or disturbances with bounded gradients [8]. It has been developed to control systems with a relative degree equal to one in order to avoid chattering.

Trajectories on the 2 sliding plane are characterized by twisting around the origin, but the continuous control law \( u(t) \) is constituted by two terms. The first one is defined by the discontinuous time derivative and the second one is a continuous function of the available sliding variable [2].

1) Controller approach: The derivative of the sliding surface is given as

\[ \dot{s} = f_e(\xi) + g_e(\xi)u + 3\lambda_1 \dot{z}(3) + 3\lambda_1 \dot{z}(2) + \lambda_1 \dot{z}(1) \]  

(30)

which can be expressed as:

\[ \dot{s} = \Psi(\xi) + \varphi(\xi)u \]  

(31)

where \( \Psi(\xi) = f_e(\xi) + 3\lambda_1 \dot{z}(3) + 3\lambda_1 \dot{z}(2) + \lambda_1 \dot{z}(1) \). The total controller can be expressed by [8]:

\[ u = u_{eq} - \Psi(\xi) \]  

(32)

where the super twisting controller

\[ u_{eq} = -k_1 \text{sign}(s)|s|^{1/2} - k_2 s + \sigma \]  

(33)

Variations of the term \( \sigma \) are described by:

\[ \dot{\sigma} = -k_3 \text{sign}(s)|s| \]  

(34)

where \( k_1, \ldots, k_4 \) are positive scalars.

The substitution of (32) and (33 into (31) gives:

\[ \dot{s} = -k_1 \text{sign}(s)|s|^{1/2} - k_2 s + \sigma \]  

(35)

2) Stability study: For the stability proof, the Lyapunov function candidate given in [10] is used:

\[ V_2(s, z) = 2k_3|s| + k_3 \sigma^2 + \frac{k_5}{2} \sigma^2 + \gamma^2 \]  

(36)

where \( k_5 \) a positive scalar and

\[ \gamma = k_1 \text{sign}(s)|s|^{1/2} + k_2 s - \sigma. \]  

(37)

We have introduced the positive scalar \( k_5 \) for a more flexibility of the stability conditions and for more generality of the expression of \( V_2 \). The substitution of (37) into (36) gives:

\[ V_2(s, \sigma) = 2k_3|s| + k_3 \sigma^2 + \frac{k_5}{2} \sigma^2 + \left( k_1 \frac{s}{\sqrt{|s|}} + k_2 s - \sigma \right)^2 \]
\[ = \frac{1}{2|s|} (4k_3|s|^2 + 2k_5 \sigma^2|s| + k_5 \sigma^2|s| + 2k_5 \sigma^2) \]
\[ + 4k_1 k_2 \sigma^2 \sqrt{|s|} - 4k_1 \sigma \sqrt{|s|} + 2k_5 \sigma^2|s| \]
\[ - 4k_2 s|\sigma + 2\sigma^2|s| \]  

(38)

Define the subspace

\[ \kappa = \{(s, \sigma) \in \mathbb{R}^2 / s = 0\} \]  

(39)

Differentiating with respect to time (38) gives:
\[ V_2(s, \sigma) = \frac{V_2}{s} ds + \frac{V_2}{ds} d\sigma \]
\[ = \frac{V_2}{ds} s \frac{V_2}{d\sigma} \sigma \]
\[ = \frac{-s}{|s|^{\frac{3}{2}}} \left( -2k_1 |s|^{\frac{3}{2}} \text{sign}(s) - 2k_4 |s|^{\frac{3}{2}} \right) \]
\[ - 2k_1^2 |s|^2 - 4k_1 k_2 |s|^3 + k_1 k_2 s^2 |s| \text{sign}(s) \]
\[ + 2k_1 |s|^2 - k_1 \sigma |s|^2 s - 2k_2^2 |s|^2 \]
\[ + 2k_2 |s|^2 + k_2^2 s^2 \text{sign}(s) \sqrt{|s|} \]
\[ - \frac{s}{|s|} \left( -k_5 |s| + 2k_1 s \sqrt{|s|} + 2k_2 |s| - 2\sigma |s| \right) \]

The closed loop is stable if matrices \( \Psi \) and \( \Upsilon \) are positive definite.
Matrix \( \Psi \) is positive definite if:

\[
\begin{cases}
\Psi_{11} > 0 \\
\Psi_{22} > 0 \\
\det(\Psi) > 0
\end{cases}
\]  

These conditions are fulfilled if:

\[
-16k_1^4 - 5k_3 k_5^2 + 20k_1^2 k_3 k_5 > 0
\]  

that is to say if:

\[
- \left( 4k_1^2 - \frac{5}{2} k_3 k_5 \right)^2 + \frac{5}{4} k_3 k_5 > 0
\]  

it is obvious that matrix \( \Psi \) is positive definite if

\[ 4k_1^2 = \frac{5}{2} k_3 k_5 \]

Define \( X = (|s|^{1/2} s \sigma)^T \). Then, it is easy to show that:

\[
\dot{V}_2(s, \sigma) \leq - \frac{1}{|s|^{1/2}} X^T \Psi X - X^T \Upsilon X
\]  

where

\[
\Psi = \begin{bmatrix} \Psi_{11} & \Psi_{13} \\ 0 & \Psi_{22} \\ \Psi_{31} & \Psi_{23} \end{bmatrix}, \quad \Upsilon = \begin{bmatrix} \Upsilon_{11} & \Upsilon_{13} \\ 0 & \Upsilon_{22} \\ \Upsilon_{31} & \Upsilon_{23} \end{bmatrix}
\]

(43) These conditions are carried out if:

\[
-k_2^2 k_3^2 + 8k_2^2 k_3 k_5 - 2k_2^4 > 0
\]  

namely if:

\[
-2 \left( k_2^2 - 2k_3 k_5 \right)^2 + 7k_3^2 k_5^2 > 0
\]  

then that matrix \( \Psi \) is positive definite if we choose

\[ k_2^2 = 2k_3 k_5 \]

So the matrices \( \Psi \) and \( \Upsilon \) are positive definite and consciously \( \dot{V}_2(s, \sigma) \leq 0 \). Thus, we can conclude that the system is stable.
IV. SIMULATION RESULTS

Parameters of the inverted pendulum system are set as: \( M = 20 \text{ kg}, \ m_0 = 1.8 \text{ kg}, \ l = 0.3 \text{ m}, \ g = 9.8. \)

Initial conditions of the cart pendulum are \((y_0, \dot{y}_0) = (0, 0), \ (\theta_0, \dot{\theta}_0) = (0.1, 0)\) and the desired position are set as \( y_d = 2, \ \theta_d = 0 \) and \( \dot{y}_d = \dot{\theta}_d = 0. \)

Simulations are done using: \( \lambda = 1 \) and \( k = 20 \) for the SMC, \( k_1 = 40 \) and \( k_2 = 90 \) for the twisting controller.

![Fig. 2. Evolution of the position of \( \theta \) for the uncertain system](image)

![Fig. 3. Evolution of the position of \( y \) for the uncertain system](image)

In Figs. 2 and 3, simulation results for the three controllers have been done. The convergence of state variables have been established for all controllers. Furthermore, state variables for STW controller converge faster than those of TW and SMC. In this simulation, 20% of mass uncertainties, have been considered for the pendulum and cart. We can notice the robust behavior of the controllers with respect to parametric uncertainties Figs. 4, 5 and 6 show that the proposed SSMC is able to compensate effectively the chattering phenomenon better than the first order sliding mode. Moreover, with the super-twisting controller the chattering is eliminated.

V. CONCLUSION

In this paper, a second order sliding mode controller SSMC has been designed for underactuated manipulators. This controller keeps main advantages of the original sliding mode approach, and removes the chattering caused by the sliding mode approach. Simulation results of the twisting and the super-twisting controllers show that the proposed controller give better performances compared to the first order sliding mode controller. It has been shown that the new sliding surface for the twisting controller reduces the chattering phenomenon. Moreover, the second-order sliding-mode controller is an effective tool for the control of uncertain nonlinear systems since
it overcomes main drawbacks of the classical sliding-mode control approach

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