Modelling and Control of Water Tank Model

Jiri Vojtesek and Petr Dostal

***Abstract*—**The paper is focused on the description of the procedure from the modelling and simulation to the adaptive control of model of the water tank as a part of the Process Control Teaching system PCT40. First, the mathematical model of the water tank is derived with the use of material balance inside and the resulting nonlinear ordinary differential equation is solved numerically with the use of the mathematical software Matlab. Results from the steady-state and dynamic analyses are then used for the choice of the optimal working point and the choice of the External Linear Model for the control. The adaptive approach here uses polynomial approach, recursive identification and pole-placement method with spectral factorization. Resulted controller has one tuning parameter – position of the root inside the closed loop and the choice of this root affects mainly the speed of the control and the overshoots.

***Keywords*—**Water Tank, Adaptive Control, Mathematical Model, Pole-placement Method, Recursive Identification.

I. Introduction

THE modelling and simulation is of the first step before the choice of the optimal working point and control strategy. These days, when computation power and speed of personal or industrial computers are very high and the price is low the role of the simulation grows.

The system could be described either mathematically or practically [1], [2]. The mathematical description for example uses material, heat etc. balances [3] depending on the type of the system, whether it is chemical reactor (Russell and Denn 1972), heat exchanger or electric motor. On the other hand, real model is usually small representation of the originally nonlinear system and we expect that results of experiments on this model are also valid or comparable to those on the real system. The big advantage of the mathematical modelling is in its safety – experiments on some real systems could be sometime hazardous. Nevertheless, experiments on the real or abstract model are usually much cheaper than those on the original system which is sometimes big and components are expensive.

This goal of this contribution is to describe how the simulation could help us with the designing of the controller for real model of the water. This real model is represented here by the Armfield's Process Control Teaching System PCT40 [4] which has several process control models and one of them is the water tank.

The mathematical model of this water tank system is mathematically described by the first order nonlinear Ordinary Differential Equation (ODE) [1]. This mathematical model is then subtracted to static and dynamic analyses.

The static analysis means solving of this ODE in the steady-state, i.e. the derivatives with the respect to time are equal to zero [3]. The nonlinear ODE is then reduced to the nonlinear algebraic equation which can be solved for example with the use of simple iteration methods [5]. The result of the static analysis could be optimal operating point or the range where the input variable could vary from the practical point of view.

On the other hand, the dynamic analysis observes the behavior of the system after the step change of the input quality, in this case the change of the feed volumetric flow rate inside the water tank. The dynamic analysis means mathematically the use one of numerical methods for solving of the ODE. The main groups of numerical methods are one-step methods for example Euler's method, Runge-Kutta's method, or multi-step methods Predictor-Corrector etc. [6]. The advantage of these methods is that they are easily programmable even more they are build-in functions in the mathematical software like Matlab [7], Mathematica etc. [8].

The adaptive control [9] approach here is based on the choice of the External Linear Model (ELM) of the originally nonlinear system [10], parameters of which are estimated recursively and the control design employs polynomial approach with pole-placement method and spectral factorization. These methods satisfies basic control requirements such as stability, disturbance attenuation and reference signal tracking.

The ELM here uses so called delta-models [11] which are special types of discrete-time models parameters of which are related to the sampling period which means that these parameters approaches to the continuous ones if the sampling period is adequately small.

The on-line identification was realized by the Recursive Least-Squares (RLS) Method which is simple, easily programmable method and if we combine this method with some kind of forgetting, for example exponential or directional, it provides satisfactory results.

All methods are verified by simulations in the mathematical software Matlab, version 7.0.1.

J. Vojtesek is with the Department of Process Control, Tomas Bata University in Zlin, nam. TGM 5555, 76001 Zlin, Czech Republic (corresponding author to provide phone: +420576035199; fax: +420576032716; e-mail: vojtesek@fai.utb.cz).

P. Dostal is with the Department of Process Control, Tomas Bata University in Zlin, nam. TGM 5555, 76001 Zlin, Czech Republic (e-mail: dostalp@fai.utb.cz).

II. Model of the Water Tank

The equipment under the consideration is the real model of the water tank which is one part of the Multifunctional process control teaching system PCT40 from Armfield [4] – see Fig. 1. The PCT40 includes also other models of processes such as Continuous Stirred Tank Reactor (CSTR) or heat exchanger.

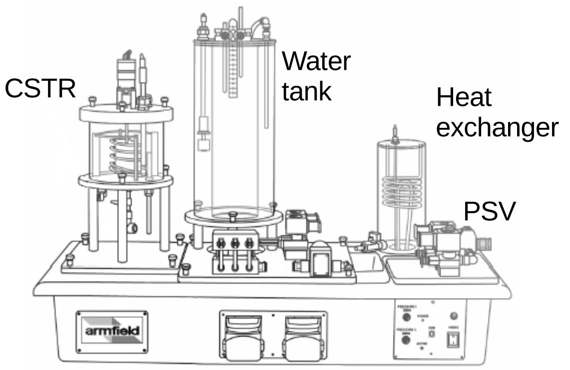

Fig. 1 Multifunctional process control teaching system PCT40

This system combines both modelling techniques – it is small representation of the water tank with the volume of 4-liter original of which is usually much bigger with huge volume. The mathematical model of this system could be also easily derived. The schematic representation of the water tank can be found in Fig. 2.

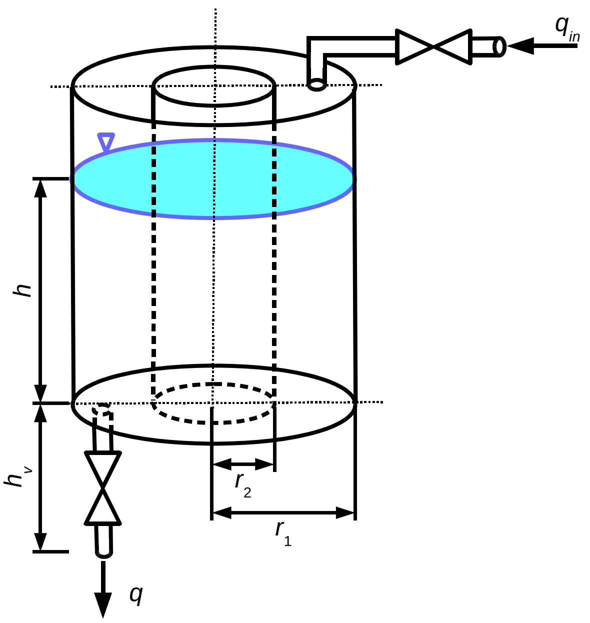

Fig. 2 Schematic representation of the water tank

The model consists of plastic transparent cylinder with inner radius $r_1 = 0.087\ m$. There is another plastic transparent cylinder inside due to quicker dynamic response of the system lower usage of feeding water. The outer radius of this smaller cylinder is $r_2 = 0.057\ m$ and the maximal water level in the tank is $h_{max} = 0.3\ m$.

In the Fig. 2, q denotes the volumetric flow rate, h is used for the water level and r are radiuses of inner and outer cylinders. The input variable is the volumetric flow rate of the feeding water q_{in} and state variables are water level h in the tank and output volumetric flow rate of the water which comes from the tank, q.

The mathematical model comes from the material balance inside the tank which is in the word form [3]:

$$\boxed{\text{Flow rate } \textbf{into the system}} = \boxed{\text{Flow rate } \textbf{out of the system}} + \boxed{\text{Rate of } \textbf{accumulation}}$$

and mathematically:

$$q_{in} = q + \frac{dV}{dt} \tag{1}$$

where V is a volume of the water inside the tank and t is used for the time.

The volume of the tank is generally

$$V = F \cdot h \tag{2}$$

for F as a area of the base due to cylindrical shape of the tank. It means, that balance (1) could be rewritten to the form

$$q_{in} = q + F \cdot \frac{dh}{dt} \tag{3}$$

where F is in this case

$$F = \pi \cdot r_1^2 - \pi \cdot r_2^2 = 1.36 \cdot 10^{-2}\, m^2 \tag{4}$$

It is also known, that volumetric flow rate through the water valve is nonlinear function of the water level, i.e.

$$q = k \cdot \sqrt{h} \tag{5}$$

where k is a valve constant which is specific for each valve and depend on the geometry and type of the valve.

If we put equation (5) inside (3) the resulting mathematical model is:

$$\frac{dh}{dt} = \frac{q_{in} - k \cdot \sqrt{h}}{F} \tag{6}$$

There should be introduced one simplification – the height of the discharging valve, h_v in Fig. 2, is neglected.

The unknown constant k could be computed for example from the steady state (variables with superscript $(\cdot)^s$), where $q_{in}^s = q^s$ and equation (5) is

$$q^s = k \cdot \sqrt{h^s} \Rightarrow k = \frac{q^s}{\sqrt{h^s}} \tag{7}$$

The water tank is fed via Proportioning Solenoid Valve (PSV) which could be operated in the range 0 – 100%. This range is from the practical point of view limited to the range $0 - 2.5 \cdot 10^{-5}\ m^3.s^{-1}$.

The equation (7) have still one unknown – the constant of the outlet valve, k. This constant could be computed from the reference measurement where we know the input volumetric flow rate, q_{in}, and after some time also the steady-state value of the water level, h^s. For example, we made the measurement for the step change from 0 to 60% of the volumetric flow rate and the results are shown in Fig. 3 – solid line.

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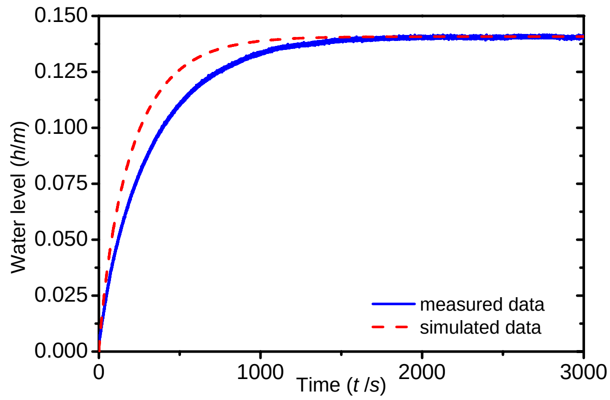

Fig. 3 Measured and simulated data for $q_{in} = 1.5 \cdot 10^{-5}$ $m^3.s^{-1}$ – computation without the valve

The volumetric flow rate is in this case $q_{in} = 1.5 \cdot 10^{-5}$ $m^3.s^{-1}$ and the final (steady-state) value of the water level h is $h^s = 0.141$ m. It means, that the valve constant k is

$$k = \frac{q_{in}}{\sqrt{h^s}} = \frac{1.5 \cdot 10^{-5}}{\sqrt{0.141}} = 4.01 \cdot 10^{-5} \tag{8}$$

The simulation is very often connected to the verification part because it is good to know if the derived mathematical model is accurate enough.

The result of the first simulation analysis for the same input volumetric flow rate $q_{in} = 1.5 \cdot 10^{-5}$ $m^3.s^{-1}$ is shown in Fig. 3 – the dashed line. It is clear, that although simulated and measured outputs reaches the same final value, the dynamics is much different – the mathematical model has quicker output response. This statement means that the mathematical model (6) is not accurate and we must neglect some simplifications. In this case we did not take into the account the height of the valve, h_v = 0.076 m, which has also impact to the mathematical model of the system.

If we count with this height, new constant of the valve for the same step change as in previous case is

$$k = \frac{q_{in}}{\sqrt{h^s}} = \frac{1.5 \cdot 10^{-5}}{\sqrt{(0.141 + 0.076)}} = 3.22 \cdot 10^{-5} \tag{9}$$

The comparison of the measured and simulated data for this new constant of the valve is shown in Fig. 4.

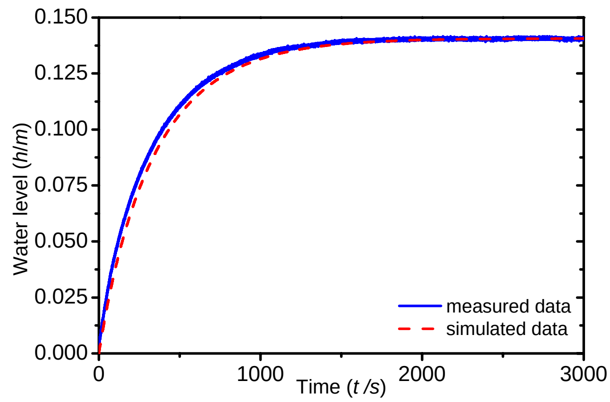

Fig. 4 Measured and simulated data for $q_{in} = 1.5 \cdot 10^{-5}$ $m^3.s^{-1}$ – computation with the valve

A. Steady-state Analysis.

The steady-state analysis means that we solve the mathematical model with the condition $d(\cdot)/dt = 0$, i.e. ODE (6) is transferred to the nonlinear algebraic equation:

$$h^s(q_{in}) = \left(\frac{q_{in}}{k}\right)^2 \tag{10}$$

where the optional variable is the input volumetric flow rate, q_{in}. There were done simulation analysis for the range q_{in} = <0; $2.5 \cdot 10^{-5}$> $m^3.s^{-1}$ and results are shown in the Fig. 5.

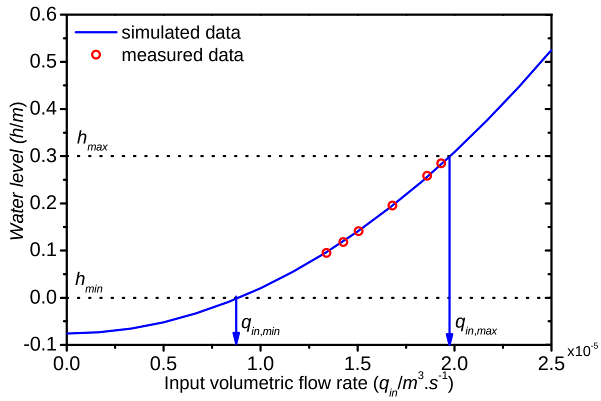

Fig. 5 The steady-state analysis of the mathematical model

This analysis shows nonlinear behavior of the system and also we can choose the volumetric flow rate in the range q_{in} = <$8.86 \cdot 10^{-6}$; $1.98 \cdot 10^{-5}$> $m^3.s^{-1}$ because lower value of q_{in} means that we did not get enough water in the tank and vice versa – the flow rate bigger than $q_{in} = 1.98 \cdot 10^{-5}$ $m^3.s^{-1}$ results in bigger water level than its maximal value h_{max}. Dots in the Fig. 5 display results of measured steady-state values.

B. Dynamic Analysis.

The dynamic analysis solves the ODE with the use of some numerical methods. In this case, the Runge-Kutta's standard method was used because it is easily programmable and even more it is build-in function in used mathematical software Matlab. The working point was characterized by the input volumetric flow rate $q_{in}^s = 1.5 \cdot 10^{-5} m^3 \cdot s^{-1}$ which is in the middle of the operating interval defined after the static analysis in the Fig. 5.

The input variable, $u(t)$, is the change of the initial q_{in}^s in % and the output variable is the water level in the tank. The input and the output variables are then generally:

$$u(t) = \frac{q_{in}(t) - q_{in}^s}{q_{in}^s} \cdot 100\,[\%];\ y(t) = h(t)\,[m] \tag{11}$$

The simulation time was 3000 s, six step changes of the input variable $u(t)$ were done and results are shown in Fig. 6.

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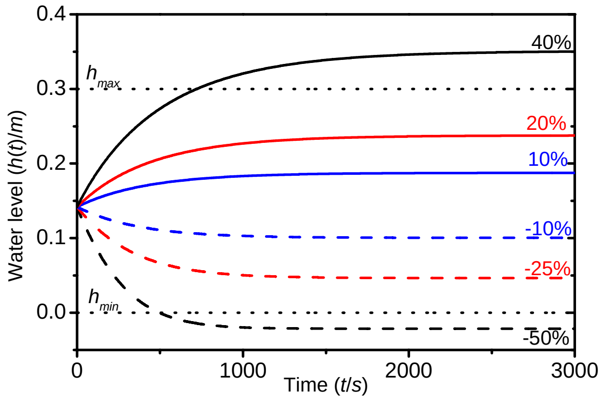

Fig. 6 The dynamic analysis for various step changes of the input volumetric flow rate q_{in}

Output responses show that this output has asymmetric responses – the final value is different in sign and also in order for positive and negative step changes. Even more, for it is inappropriate to choose the input step change of the volumetric flow rate lower than approximately -40% and bigger than +30% because the resulted water level is lower or higher than physical properties of the water tank.

III. Adaptive Control

The adaptive control was used for control of this system. There are several adaptive approaches which can be used. The method here uses External Linear Model (ELM) as a linear description of the originally nonlinear system. Parameters of and the structure of the controller are derived from this ELM and its parameters are identified recursively during the control. Parameters of the controller are recomputed in each time period too which means that this controller adopts its parameters according to the actual state and behavior of the controlled system.

In this case, all output responses in Fig. 6 could be expressed by the first or the second order transfer functions (TF), for example in the continuous-time

$$G_1(s)=\frac{b(s)}{a(s)}=\frac{b_0}{s+a_0}$$
$$G_2(s)=\frac{b(s)}{a(s)}=\frac{b_1 s+b_0}{s^2+a_1 s+a_0} \tag{12}$$

A. Control System Synthesis

The controller is constructed with the use of polynomial synthesis and the control structure with 1DOF is shown in Fig. 7.

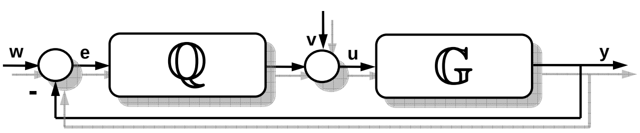

Fig. 7 1DOF control configuration

The block G denotes transfer function (12) of controlled plant, w is the reference signal (wanted value), v is disturbance, e is used for control error, u is control variable and y is a controlled output. The transfer function of the feedforward part $Q(s)$ of the controller is designed with the use of polynomial synthesis:

$$\tilde{Q}(s)=\frac{q(s)}{s\cdot\tilde{p}(s)} \tag{13}$$

where degrees of polynomials $\tilde{p}(s)$ and $q(s)$ are computed from:

$$\begin{aligned}&\deg q(s)=\deg a(s)+\deg f(s)-1\\&\deg\tilde{p}(s)\geq\deg a(s)-1\end{aligned} \tag{14}$$

and parameters of these polynomials are computed by the Method of uncertain coefficients which compares coefficients of individual s-powers from the Diophantine equation, e.g. [12]:

$$a(s)\cdot s\cdot\tilde{p}(s)+b(s)\cdot q(s)=d(s) \tag{15}$$

and the polynomial $d(s)$ on the right side of (15) is an optional stable polynomial. It is obvious, that the degree of this polynomial is:

$$\deg d(s)=\deg a(s)+\deg\tilde{p}(s)+1 \tag{16}$$

Roots of this polynomial are called poles of the closed-loop and their position affects quality of the control.

This polynomial could be designed for example with the use of Pole-placement method. A choice of roots needs some a priory information about the system's behavior. It is good to connect poles with the parameters of the system via spectral factorization. The polynomial $d(s)$ can be then rewritten to the form

$$d(s)=n(s)\cdot(s+\alpha_i)^{\deg d-\deg n} \tag{17}$$

where $\alpha_i > 0$ is an optional coefficient reflecting closed-loop poles and stable polynomial $n(s)$ is obtained from the spectral factorization of the polynomial $a(s)$

$$n^*(s)\cdot n(s)=a^*(s)\cdot a(s) \tag{18}$$

The Diophantine equation (15), as it is, is valid for step changes of the reference and disturbance signals which means that deg $f(s) = 1$ in (14). The feedback controller $Q(s)$ ensures stability, load disturbance attenuation and asymptotic tracking of the reference signal.

B. External Linear Model (ELM)

The ELM here comes from the dynamic analysis as it is written above. The TF in (12) belongs to the class of continuous-time (CT) models. The identification of such processes is not very easy.

One way, how we can overcome this problem is the use of so called δ–model. This model belongs to the class of discrete models but its parameters are close to the continuous ones for very small sampling period as it proofed in [13].

The δ–model introduces a new complex variable γ, for example

$$\gamma=\frac{z-1}{T_v} \tag{19}$$

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If we choose for simplification first order TF G_1 in (12), the differential equation will be

$$y_\delta(k) = b_0^\delta u_\delta(k-1) - a_0^\delta y_\delta(k-1) \tag{20}$$

where b_0^δ and a_0^δ are delta parameters different from the parameters b_0 and a_0 in (12) and the individual parts in Equation (20) can be written as

$$y_\delta(k) = \frac{y(k)-y(k-1)}{T_v}; \qquad u_\delta(k-1) = u(k-1) \tag{21}$$
$$y_\delta(k-1) = y(k-1);$$

The regression vector $\boldsymbol{\varphi}_\delta$ is then

$$\boldsymbol{\varphi}_\delta(k-1) = \left[-y_\delta(k-1), u_\delta(k-1)\right]^T \tag{22}$$

and the vector of parameters $\boldsymbol{\theta}_\delta$ is generally

$$\boldsymbol{\theta}_\delta(k) = \left[a_1^\delta, a_0^\delta, b_1^\delta, b_0^\delta\right]^T \tag{23}$$

which is computed from the differential equation

$$y_\delta(k) = \boldsymbol{\theta}_\delta^T(k) \cdot \boldsymbol{\varphi}_\delta(k-1) + e(k) \tag{24}$$

where $e(k)$ is a general random immeasurable component.

As it is written in the previous part, control system synthesis is done in continuous time, but recursive identification uses discrete time steps. The resulted, so called "hybrid", controller works in the continuous time but parameters of the polynomials $a(s)$ and $b(s)$ are identified recursively in the sampling period T_v. This assumption results in the condition, that the parameters of the δ-model are close the continuous ones for the small sampling period.

IV. Simulation Results

Simulation analyses do the control exercises on the mathematical model of the water tank (6) where the reference signal (wanted value) is the level of the water in the tank, h, which is controlled by the change of the input volumetric flow rate q_{in}.

The sampling period was $T_v = 1$ s, the simulation time was 5000 s and 5 different step changes of the reference signal was done during this time. The controller could be tuned with the choice of the parameter α_i. The affect of this parameter are shown in following figures.

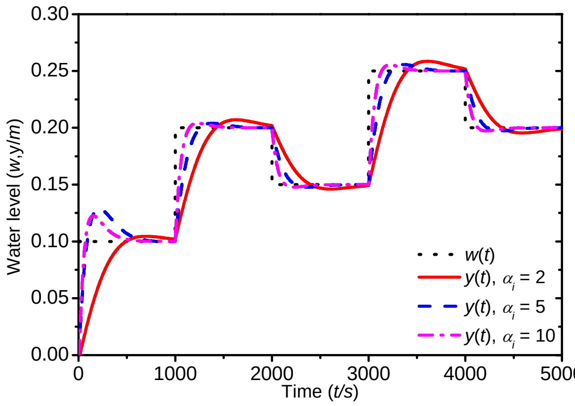

Fig. 8 The course of the reference signal, $w(t)$, and the output variable, $y(t)$, for different values of α_i

The Fig. 8 clearly shows the effect of the tuning parameter α_i – increasing value of this parameter results in quicker output response with the overshoots. The output response for the lowest value, i.e. $\alpha_i = 2$, produces more smoother course of the output variable without the overshoot at the very beginning in the first step change.

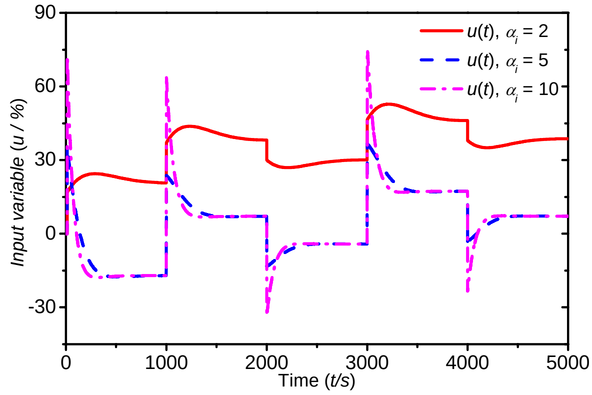

Fig. 9 The course of the input variable, $u(t)$, for different values of α_i

The course of the input variable on the other side is very similar for α_i = 5 and 10. The third course for α_i = 2 is different and smoother on the contrary.

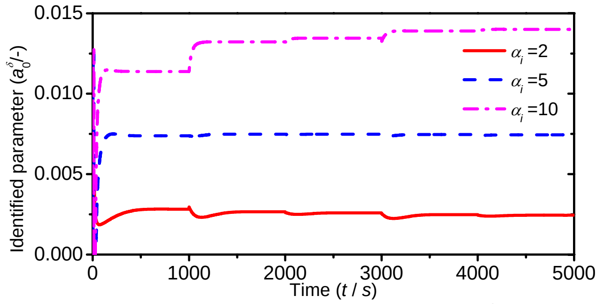

Fig. 10 The course of the identified parameter a_0^δ for different values of α_i

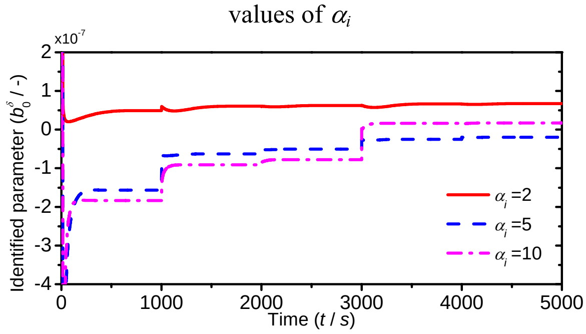

Fig. 11 The course of the identified parameter b_0^δ for different values of α_i

Last graphs in Fig. 10 and Fig. 11 shows values of the estimated parameters a_0^δ and b_0^δ. It is clear, that used recursive identification has not problem with the on-line identification except the very beginning of the control, which is caused by the uncertainty of the system which needs some time for estimation of the real parameters of the system. This is typical feature of this type of adaptive control.

V. CONCLUSION

The goal of this contribution is to show one way how to design the controller for the real system. At first, the mathematical model with one ordinary differential equation is derived. This model was then verified by the simulations of the steady-state and dynamics and the results are compared with the measurements on the real system. This comparison shows disproportion between simulated and measured data which is caused by the inaccuracy of the model which does not take into the account the height of the outlet valve. If we include this height into the computation, the results are much more accurate. The control approach here is based on the choice of the external linear model of the originally nonlinear system, parameters of which are identified recursively. The simple controller with one degree-of-freedom is designed with the use of polynomial approach with pole-placement method and spectral factorization. The resulted controller is stable and satisfies basic control requirements. Moreover, it can be tuned by the choice of the parameter α_i – increasing value of this parameter results in quicker output response but possible overshoots. Introduced hybrid adaptive controller produces good control results although the system has nonlinear behavior. The future work is connected with the verification of the proposed control strategy on the real model of the water tank.

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