The influence of variable friction coefficient on spatial stability of slightly curved shallow mixing layers

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***Abstract*—**Spatial stability analysis is used in the present paper in order to investigate the effect of variable friction on spatial growth rates of an unsteady perturbation. The flow is assumed to be slightly curved in the longitudinal direction. Numerical scheme is developed for the solution of the spatial stability problem. Numerical results show that growth rate of a perturbation decreases in the presence of regions of non-uniform friction. In addition, small curvature stabilizes the flow.

***Keywords*—**Spatial stability, rigid-lid assumption, shallow mixing layers, variable friction.

I. Introduction

Stability of shallow mixing layers is investigated in several papers [1]-[4]. Rigid-lid assumption is used in [1] to analyze linear stability of mixing layers and wakes in shallow water. The role of Froude number on the stability boundary is studied in [2] where it is shown that rigid-lid assumption works well (taking into account linear stability characteristics of the flow) for small Froude numbers. Gravitational and shear instabilities in compound and composite channels are analyzed in [3]. Linear stability analysis is one of the widely used methods for the analysis of shallow flows. Other methods include experimental investigation and numerical simulations [4].

Shallow mixing layers are also analyzed experimentally in [5]-[7]. It is shown in [5]-[7] that limited water depth in shallow flows has a stabilizing influence on the flow. Mixing layer width also decreases downstream in contrast to the case of unbounded mixing layers.

Recently an important practical problem is investigated experimentally in a series of papers [7]-[11] published by MIT group. The authors analyzed shallow mixing layers in the presence of a porous layer which can be formed, for example, by aquatic vegetation. Such flows can occur during floods. In this case friction coefficient of the flow is not constant (as it is assumed, for example, in [1]-[4]), but varies in the transverse direction of the flow. Linear stability analysis of shallow mixing layer under the assumption that the friction coefficient is represented by a step function is performed in [7]. Later (see, for example, [12]) linear stability of shallow mixing layers is investigated for the case where the friction coefficient varies continuously with respect to the transverse coordinate from zero to some fixed constant.

In the present paper we analyze spatial stability of shallow mixing layers in compound channels for the case where two additional factors are considered: (a) the flow is assumed to be slightly curved in the longitudinal direction and (b) the friction coefficient of the flow varies continuously with respect to the transverse direction. The hyperbolic tangent function is used to model bottom friction. In addition, the friction coefficient approaches non-zero constant values (smaller value in the main channel and larger value in the floodplain). Numerical results are presented for different values of the parameters of the problem.

II. Mathematical Formulation of the Problem

We consider the system of shallow water equations under the rigid-lid assumptions of the form

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{\partial p}{\partial x}+\frac{c_f(y)}{2h}u\sqrt{u^2+v^2}=0, \tag{2}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}-\frac{1}{R}u^2+\frac{\partial p}{\partial y}+\frac{c_f(y)}{2h}v\sqrt{u^2+v^2}=0, \tag{3}$$

where u and v are the velocity components, $c_f(y)$ is the friction coefficient, p is the pressure, h is water depth and R is the dimensionless radius of curvature ($R>>1$).

Introducing the stream function $\psi(x,y,t)$ by the relations

$$u=\frac{\partial \psi}{\partial y},\ v=-\frac{\partial \psi}{\partial x} \tag{4}$$

and eliminating the pressure from (2) and (3) we obtain

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$$(\Delta\psi)_t+\psi_y(\Delta\psi)_x-\psi_x(\Delta\psi)_y+\frac{2}{R}\psi_y\psi_{xy}+$$
$$+\frac{c_f(y)}{2h}\Delta\psi\sqrt{\psi_x^2+\psi_y^2}+\frac{c_f(y)}{2h\sqrt{\psi_x^2+\psi_y^2}}(\psi_y^2\psi_{yy}+ \quad (5)$$
$$+2\psi_x\psi_y\psi_{xy}+\psi_x^2\psi_{xx})+\frac{c_{fy}(y)}{2h}\psi_y\sqrt{\psi_x^2+\psi_y^2}=0,$$

where $c_{fy}(y)=c_{f_0}\gamma'(y)$ is the derivative of the function $c_f(y)$ with respect to y. It is assumed here that the dependence of the friction coefficient on the transverse coordinate y is given by the formula

$$c_f(y)=c_{f_0}\gamma(y), \qquad (6)$$

where $\gamma(y)$ is sufficiently smooth "shape" function.

Using the method of small perturbations we represent the solution $\psi(x,y,t)$ in the form

$$\psi(x,y,t)=\psi_0(y)+\varepsilon\psi_1(x,y,t)+... \qquad (7)$$

where $\psi_0(y)$ is the stream function of the base flow $U(y)$ such that $U(y)=\psi_{0y}(y)$. The function $U(y)$ is usually assumed to be of the form a hyperbolic tangent function. In this study, we assume that

$$U(y)=\frac{1}{2}(1+\tanh y). \qquad (8)$$

Substituting (6) into (5) and linearizing the resulting equation in the neighborhood of the base flow (7) we obtain

$$\psi_{1xxt}+\psi_{1yyt}+\psi_{0y}(\psi_{1xxx}+\psi_{1yyx})-\psi_{0yyy}\psi_{1x}+$$
$$+\frac{c_f(y)}{2h}(\psi_{0y}\psi_{1xx}+2\psi_{0yy}\psi_{1y}+2\psi_{0y}\psi_{1yy}) \qquad (9)$$
$$+\frac{c_{fy}(y)}{h}\psi_{0y}\psi_{1x}+\frac{2}{R}\psi_{0y}\psi_{1xy}=0.$$

In accordance with the method of normal modes a small unsteady perturbation of the stream function is assumed to be of the form

$$\psi_1(x,y,t)=\varphi(y)e^{i(\alpha x-\beta t)}, \qquad (10)$$

where α and β, in general, are complex. Substituting (10) into (9) we obtain

$$\varphi_{yy}(\alpha U-\beta-iSU\gamma)-iS(\gamma U_y+\gamma_y U)\varphi_y$$
$$+\varphi(\alpha^2\beta-\alpha^3U-\alpha U_{yy}+i\alpha^2US\gamma/2)=0, \qquad (11)$$

where $S=\frac{c_{f_0}b}{h}$ is the bed friction number (see [1]) and b is a characteristic length scale of the problem (for example, mixing layer half-width).

The boundary conditions are

$$\varphi(\pm\infty)=0. \qquad (12)$$

There are two basic approaches to the solution of (11), (12). The first (known as temporal stability analysis) is used under the assumption that α is real and $\beta=\beta_r+i\beta_i$ is complex. Thus, from a temporal stability point of view the base flow is unstable with respect to the perturbation with wave number α if $\beta_i>0$. The set of all values of the parameter S for which $\beta_i=0$ in the $(\alpha,S)-$plane gives the neutral stability curve. The second approach (known as spatial stability analysis) is based on the assumption that $\alpha=\alpha_r+i\alpha_i$ is complex and β is real. Flow (8) is said to be spatially unstable if at least one $\alpha_i<0$.

From a computational point of view temporal stability analysis is simpler since one has to solve a linear generalized eigenvalue problem while in order to analyze spatial stability a polynomial generalized eigenvalue problem has to be solved. However, many experimental observations for shallow mixing layers deal with spatial variation of the characteristics of the flow. Hence, spatial stability analysis is used in practice more often.

In the present paper eigenvalue problem (11), (12) is solved as a spatial stability problem.

III. Numerical Method

Problem (11), (12) is solved by means of a collocation method. First, the interval $-\infty<y<+\infty$ is mapped onto the interval $-1\le\xi\le 1$ by means of the transformation $\xi=\frac{2}{\pi}\arctan y$. The solution to (11) is sought in the form

$$\varphi(\xi)=\sum_{k=0}^{N-1}a_k(1-\xi^2)T_k(\xi), \qquad (13)$$

where $T_k(\xi)=\cos k\arccos\xi$ is the Chebyshev polynomial of the first kind of order k and a_k are unknown coefficients. The factor $(1-\xi^2)$ is added in (13) in order to satisfy the zero boundary conditions

$$\varphi(\pm 1)=0. \qquad (14)$$

The collocation points are

$$\xi_m=\cos\frac{\pi m}{N}, \quad m=1,2,...,N-1. \qquad (15)$$

Substituting (13) into (11) and using (15) we obtain the following generalized eigenvalue problem

$$(A+\alpha B)a=0, \qquad (16)$$

where A and B are complex-valued nonsingular matrices and $a=(a_0a_1...a_{N-1})^T$.

Problem (16) is solved numerically for different values of the parameters of the problem.

IV. Numerical Results

The variation of the friction coefficient in the transverse direction is assumed to be of the form

$$c_f(y) = c_{f_2}\left(\frac{\gamma+1}{2} + \frac{(\gamma-1)}{2}\tanh \mu y\right), \qquad (17)$$

where $\gamma = \frac{c_{f_1}}{c_{f_2}} \ge 1$ and $c_f(y) \to c_{f_1}$ as $y \to -\infty$, $c_f(y) \to c_{f_2}$ as $y \to +\infty$. Here c_{f_1} and c_{f_2} are nonzero constants. Note that in [12] the friction coefficient varied is such a way that $c_f(y) \to 0$ as $y \to -\infty$ and $c_f(y) \to c_{f_2}$ as $y \to +\infty$. In other words, a frictionless flow is assumed in [12] as $y \to -\infty$. The parameter μ in (17) represents how sharp is the transition from the region of larger friction to the region of smaller friction.

Variability of the friction coefficient given by (17) is consistent with the velocity profile (8) since higher velocity is expected in the region where friction force is smaller.

Fig. 1 plots the spatial growth rates for the case $S = 0.15, \mu = 1$ and $R = \infty$ (no curvature). The case $\gamma = 1$ (top curve) corresponds to uniform friction. As can be seen from the graph, non-uniform friction of the form (17) has a stabilizing influence on the flow: the growth rate for the most unstable mode decreases as the parameter γ increases. Note that γ represents the degree of non-uniformity of the friction force in the transverse direction. Thus, flow with non-uniform friction is more stable than flow with uniform friction.

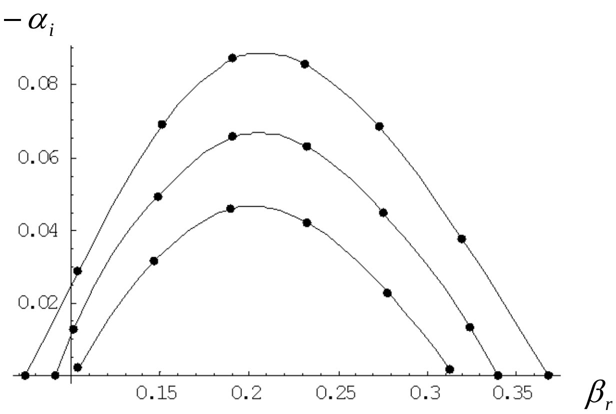

Fig. 1. Spatial growth rates $-\alpha_i$ versus β_r for three values of $\gamma : \gamma = 1, \gamma = 1.5$ and $\gamma = 2$ (from top to bottom).

The role of curvature on the stability characteristics of the flow is seen from Fig. 2 where the spatial growth rate for the case $S = 0.15$ and $\gamma = 1.5$ is shown for three values of the parameter $1/R$, namely, $1/R = 0$ (straight flow with no curvature), 0.01 and 0.02.

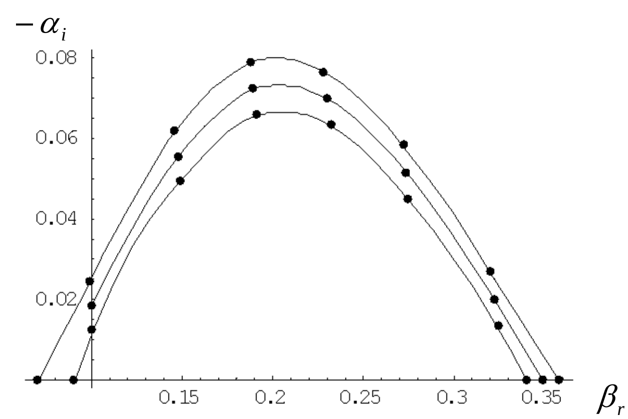

Fig. 2. Spatial growth rates $-\alpha_i$ versus β_r for three values of $1/R : 0, 0.01$ and 0.02 (from bottom to top).

The bottom curve in Fig. 2 corresponds to the case of no curvature and is the most stable among the three cases considered. Thus, increase in curvature has a destabilizing effect on the flow.

The effect of the parameter μ on the spatial growth rates is shown in Fig. 3 for the case $S = 0.15, \gamma = 2, 1/R = 0$.

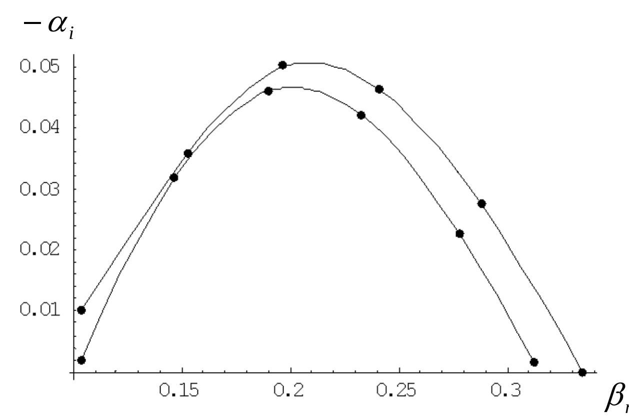

Fig. 3. Spatial growth rates $-\alpha_i$ versus β_r for two values of $\mu : 1$ and 10 (from bottom to top).

It is seen from Fig. 3 that steeper friction gradients result in less stable flow.

V. Conclusion and Direction of Future Work

Spatial linear stability analysis of shallow mixing layers is performed in the present paper. The effect of several parameters of the stability characteristics of the flow is investigated. In particular, it is shown that non-uniform friction coefficient in the transverse direction of the flow has a stabilizing influence in comparison with the case of a uniform friction. In addition, numerical computations demonstrate that slightly curved mixing layers are more stable than layers without curvature. Finally, it is shown that steepness of the change of the friction coefficient in the transverse direction has a destabilizing influence on the flow.

Linear stability analysis is performed in the present paper under the assumption of a parallel flow. In other words, the base flow profile (8) is assumed to be independent on the longitudinal coordinate. Experimental data (see, for example, [5] and [6]) show that the base flow is slightly changing along the longitudinal coordinate. Asymptotic schemes have been developing in the past in order to take into account slow longitudinal variation of the base flow. The authors are currently implementing the asymptotic scheme in order to derive the amplitude evolution equation for the most unstable mode.

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