Equations of Motion and Physical Model of Quad-copter in Plain

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Abstract— Presented text documents the body quad-copter at plane with four propellers motion equations procedure creation, by force of Denavit -Hartenberg (DH) co - ordinate systems implementation and kinematic homogenous transformation matrices. Power interactions' physical model of this ordering was created and its behavior was compared with behavior of motional equations system.

Keywords— Quad-copter, Denavit-Hartenberg notation, motion equations, rigid mechanic bonded bodies, physical model, simulation experiments..

I. Introduction

Quad-copter can be seen as UAV (unmanned aerial vehicle) equipped by autonomous subsystems. it can be equipped e.g. by video subsystem, so its purpose could be the observation. But this subsystem can serve as well as the substantial part of object outside sensorial system, enabling sufficient orientation in 3D space (including distance measuring).

Because it has the capability to carry the small service load, is able to be used e.g. for small weight supply rescue deliverance to an inaccessible localities.

Nevertheless, maybe the biggest quad-copter importance is simply in skills creation at its intended complex structural project, including its movement control systems in 3D space and its independent orientation there.

Quad-copter selection motivations as top design project are different for various bands of designers.

For someone it's orientation on project and structural robot design, for others then on robot with sensors and autonomous capabilities, and eventually we meet the goal-directed orientation oriented from ground mobile robot to the flying object.

|Initially only the absorption in technical difficulties arisen, connected with flying object proposal and creation, notwithstanding flying object idea and its exploitation possibilities as well caused substantial interest, which was lastly accepted like challenge.

The main general aim of the works, one of which this text the first part, is methodology of motional equations derivation of flying object with more propellers and achieved results, necessary for sensorial system proposal and the control, comparison with physical model created by compartment access to the mentioned system as to the stiff bodies system with finite points power interaction numbers.

This text constrains, in frame of methodology derivation, only to the rotational motion in plane (yaw) object under consideration, controlled by four propellers.

Next works will be dedicated to complete flying object movement and its motional equations.

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II. Quad-copter body rotation motional equations (spin) by effect of four propellers

A. Statement of a problem

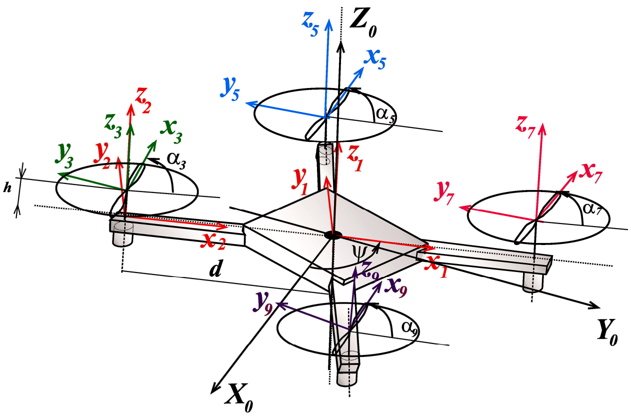

Fig. 1 Ideal quad-copter solid body in plane with four propellers and select coordinated systems

On Fig.1 quad-copter body is presented, situated in plane as a system with five freedom degrees. State quantities are quad-copter body angle of rotation ψ and its angular velocity and separate propeller fans rotation angles α_3, α_5, α_7 and α_9 and their speeds.

Task is to derive the spin motional equations depending on revolving propellers' torques.

B. Quad-copter movement dynamics

By co-ordinate systems introduction, according to DH notation in compliance with introduced picture, it is possible to write homogenous transformation matrixes, by means of single co-ordinate systems parameters, e.g. for the first propeller $\mathbf{O}_{x3y3z3}$:

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$$
{}^{2}T_{3}=\begin{bmatrix}\cos\alpha_3 & -\sin\alpha_3 & 0 & 0\\ \sin\alpha_3 & \cos\alpha_3 & 0 & 0\\ 0 & 0 & 1 & h\\ 0 & 0 & 0 & 1\end{bmatrix};\ {}^{1}T_{2}=\begin{bmatrix}1 & 0 & 0 & -d\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix};
$$

$$
{}^{0}T_{1}=\begin{bmatrix}\cos\psi & -\sin\psi & 0 & 0\\ \sin\psi & \cos\psi & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}
$$

$$
{}^{0}T_{3}=\begin{bmatrix}\cos\psi\cos\alpha_3-\sin\psi\sin\alpha_3 & -\cos\psi\sin\alpha_3-\sin\psi\cos\alpha_3 & 0 & -d\cos\psi\\ \sin\psi\cos\alpha_3+\cos\psi\sin\alpha_3 & -\sin\psi\sin\alpha_3+\cos\psi\cos\alpha_3 & 0 & -d\sin\psi\\ 0 & 0 & 1 & h\\ 0 & 0 & 0 & 1\end{bmatrix}
$$

Because mass element of the first homogeneous propeller is

$$
dm=\rho\cdot S\cdot dx=\frac{m_1}{S\cdot \ell}S\cdot dx=\frac{m_1}{\ell}\cdot dx
$$

and this element has in $\mathbf{O_{x3y3z3}}$ absolute homogenous coordinates

$$
\begin{bmatrix}x\cdot(\cos\alpha_3\cos\psi-\sin\alpha_3\sin\psi)-d\cdot\cos\psi\\ x\cdot(\cos\alpha_3\sin\psi+\sin\alpha_3\cos\psi)-d\cdot\sin\psi\\ h\\ 1\end{bmatrix}
$$

its absolute velocity vector size square is

$$
|\vec{v}|^2=\left(\dot\psi^2+\dot\alpha_3^2+2\cdot\dot\alpha_3\dot\psi\right)\cdot x^2-2\cdot d\cdot\cos\alpha_3\cdot\left(\dot\psi^2+\dot\alpha_3\dot\psi\right)\cdot x+d^2\cdot\dot\psi^2
$$

and kinetic energy of the first propeller is

$$
W_{kP1}=\frac{1}{2}\frac{m_1}{\ell}\cdot\int_{-\ell/2}^{\ell/2}\left[\left(\dot\psi^2+\dot\alpha_3^2+2\cdot\dot\alpha_3\dot\psi\right)\cdot x^2-2\cdot d\cdot\cos\alpha_3\cdot\left(\dot\psi^2+\dot\alpha_3\dot\psi\right)\cdot x+d^2\cdot\dot\psi^2\right]\cdot dx=
$$

$$
=\frac{1}{2}m_1\cdot\left(d^2+\frac{1}{12}\cdot\ell^2\right)\cdot\dot\psi^2+\frac{1}{24}m_1\cdot\ell^2\cdot\dot\alpha_3^2+\frac{1}{12}m_1\cdot\ell^2\dot\alpha_3\dot\psi
$$

Similarly for other propellers reads:

$$
W_{kP2}=\frac{1}{2}m_1\cdot\left(d^2+\frac{1}{12}\cdot\ell^2\right)\cdot\dot\psi^2+\frac{1}{24}m_1\cdot\ell^2\cdot\dot\alpha_5^2+\frac{1}{12}m_1\cdot\ell^2\dot\alpha_5\dot\psi
$$

$$
W_{kP3}=\frac{1}{2}m_1\cdot\left(d^2+\frac{1}{12}\cdot\ell^2\right)\cdot\dot\psi^2+\frac{1}{24}m_1\cdot\ell^2\cdot\dot\alpha_7^2+\frac{1}{12}m_1\cdot\ell^2\dot\alpha_7\dot\psi
$$

$$
W_{kP4}=\frac{1}{2}m_1\cdot\left(d^2+\frac{1}{12}\cdot\ell^2\right)\cdot\dot\psi^2+\frac{1}{24}m_1\cdot\ell^2\cdot\dot\alpha_9^2+\frac{1}{12}m_1\cdot\ell^2\dot\alpha_9\dot\psi
$$

Quad-copter body kinetic energy is:

$$
W_{k_{base}}=\frac{1}{2}J_{zz}\cdot\dot\psi^2
$$

Quad-copter potential energy is invariable on the movement under consideration.

Resulting motional equations are:

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\psi}\right)-\frac{\partial L}{\partial\psi}=\frac{d}{dt}\left(\frac{\partial W_k}{\partial\dot\psi}\right)-\frac{\partial W_k}{\partial\psi}=\\
&=\left[J_{zz}+4m_1\cdot\left(d^2+\frac{1}{12}\cdot\ell^2\right)\right]\cdot\ddot\psi+\frac{1}{12}m_1\cdot\ell^2\ddot\alpha_3+\\
&+\frac{1}{12}m_1\cdot\ell^2\ddot\alpha_5+\frac{1}{12}m_1\cdot\ell^2\ddot\alpha_7+\frac{1}{12}m_1\cdot\ell^2\ddot\alpha_9=-b\cdot\dot\psi\\
&\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\alpha_3}\right)-\frac{\partial L}{\partial\alpha_3}=\frac{1}{12}m_1\cdot\ell^2\cdot\ddot\alpha_3+\\
&+\frac{1}{12}m_1\cdot\ell^2\cdot\ddot\psi=M_3(t)-b_p\cdot\dot\alpha_3\\
&\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\alpha_5}\right)-\frac{\partial L}{\partial\alpha_5}=\frac{1}{12}m_1\cdot\ell^2\cdot\ddot\alpha_5+\\
&+\frac{1}{12}m_1\cdot\ell^2\cdot\ddot\psi=M_5(t)-b_p\cdot\dot\alpha_5\\
&\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\alpha_7}\right)-\frac{\partial L}{\partial\alpha_7}=\frac{1}{12}m_1\cdot\ell^2\cdot\ddot\alpha_7+\\
&+\frac{1}{12}m_1\cdot\ell^2\cdot\ddot\psi=M_7(t)-b_p\cdot\dot\alpha_7\\
&\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\alpha_9}\right)-\frac{\partial L}{\partial\alpha_9}=\frac{1}{12}m_1\cdot\ell^2\cdot\ddot\alpha_9+\\
&+\frac{1}{12}m_1\cdot\ell^2\cdot\ddot\psi=M_9(t)-b_p\cdot\dot\alpha_9
\end{aligned}
\tag{1}
$$

Matrix notation of motional equations brightly shows how it is possible to control angle $\boldsymbol{\psi}$. With respect to select orientation coming-out:

$$
\ddot\psi=-\frac{b\cdot\dot\psi+(M_3+M_5+M_7+M_9)}{J_{zz}+4\cdot m_1\cdot d^2}+\frac{b_p\cdot(\dot\alpha_3+\dot\alpha_5+\dot\alpha_7+\dot\alpha_9)}{J_{zz}+4\cdot m_1\cdot d^2}\quad(2)
$$

and so, angle $\boldsymbol{\psi}$ is control by propellers' accelerating torques.

By mentioned equations' solution quad-copter dynamic behavior can be given, e.g. at sizes spin control process by force of propellers revolution control. Stated possibility is presented on Fig. 2, where's quad-copter body turning control to one or other side, made by all propellers turning control without dissipative forces.

Is evident, that it is impossible to quad-copter control in this manner, which has to, thanks to propellers, also to levitate, eventually it has to move laterally.

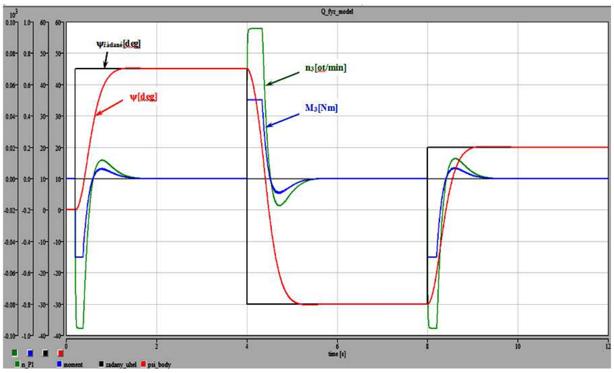

Fig. 2 Quad-copter dynamic behavior at yaw control

At total motion control will be necessary to combine various turning directions of all propellers and that way to achieve partly spin control, but also quad-copter levitation provision in a certain altitude (gravity compensation by propellers thrust) and e.g. its center of gravity lateral motion in space.

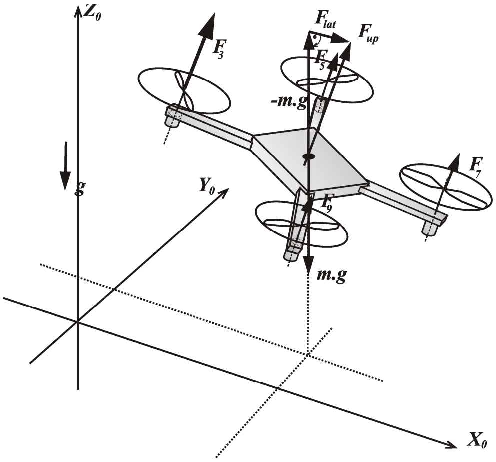

Fig. 3 Quad–copter lateral motion in axe X_0 direction.

C. Quad-copter motion physical model

Although resulting motional equations' form is simple, its derivation was labor-intensive. But it showed the universal procedure that can be used further for complete quad-copter motional equations' system in 3D space.

Next work goal is the stated arrangement physical model creation (energy interaction model) enabling simulation experiments with quad-copter rotating motion control in plane with synchronous levitation in required altitude..

1) Released body dynamics with rotary kinematic pairs

Primarily we will study the following general released body in plane with four rotational kinematic pairs according to Fig.4.

What is relation between point A local homogenous coordinates i. e. $[\mathbf{x_A}, \mathbf{y_A}]$ and its global coordinates $[\mathbf{X_A}, \mathbf{Y_A}]$?

Rotation transformation matrix around global axis Z with angle ψ is.

$$Q_{Z,\psi} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

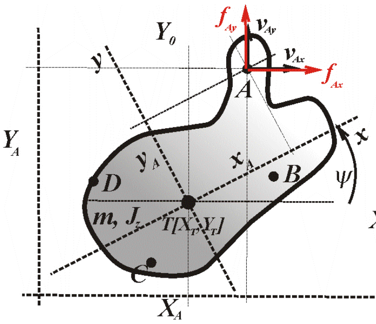

Fig. 4 General released body in plane with four rotational kinematic pairs.

So point **A** global homogenous coordinates are:

$$\begin{bmatrix} X_A(t) \\ Y_A(t) \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 & X_T(t) \\ \sin\psi & \cos\psi & 0 & Y_T(t) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_A(t) \\ y_A(t) \\ 0 \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} X_T(t) + x_A\cos\psi(t) - y_A\sin\psi(t) \\ Y_T(t) + x_A\sin\psi(t) + y_A\cos\psi(t) \\ 0 \\ 1 \end{bmatrix}$$

For its absolute velocity components (velocity in global reference coordinate system) pays:

$$\begin{bmatrix} v_{A_x}(t) \\ v_{A_y}(t) \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{dX_T(t)}{dt} - \omega(t)\cdot[x_A\sin\psi(t) + y_A\cos\psi(t)] \\ \frac{dY_T(t)}{dt} + \omega(t)\cdot[x_A\cos\psi(t) - y_A\sin\psi(t)] \\ 0 \\ 1 \end{bmatrix} \tag{3}$$

Point **A** absolute velocity is determinated by two components of across variables (velocity) $\vec{v}_A = v_{A_x}\hat{I} + v_{A_y}\hat{J}$ (where $\hat{I}, \hat{J}$ are global axes X_0 and Y_0 unitary vectors). Similarly for other points **B, C, D**.

For across variables the flow variables (generalized forces) $\vec{F}_A = f_{A_x}\hat{I} + f_{A_y}\hat{J}$ appertain (see Fig.4).

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We establish the following model of rigid body in plane

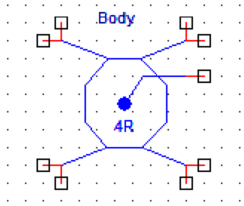

Fig. 5 Pictogram of rigid body in plane model with four rotational kinematic pairs.

Model „points of contact " are points' appropriate traversable velocity components and body angular velocity. Similarly we specify rigid body with one kinematic rotational pair model TR:

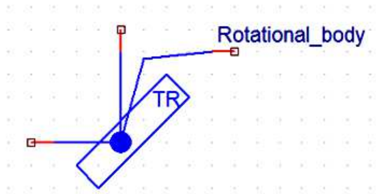

Fig. 6 Pictogram of rigid body in plane model with one rotational kinematic pair.

2) Power interactions' dynamic model

Resulting power interaction' model consists of:

1 . **Quad-copter body dynamic behavior model**, body with four rotational kinematic pairs moving with one freedom degree-angle ψ.

2. **Four times propeller's dynamic behavior model** used as stiff body with one rotational kinematic pair.

3. **Four controlled torque sources**, actuating on four propellers with the aim of achieve
   - partly quad - copter body levitation,
   - as well as its turning around quad-copter body plane normal line.

The result is achieved by the torques applied on separate propellers with forms:

$$\begin{aligned} m_{prop_3}(t) &= m_{prop_7}(t) = M_{up} + m_{reg} \\ m_{prop_5}(t) &= m_{prop_9}(t) = -M_{up} + m_{reg} \end{aligned} \tag{4}$$

where $\boldsymbol{M_{up}}$ is constant levitation torque (the propellers side by side have inverse thrust) and it's controlled torque applied on individual propellers [see equation (2)].

Resulting layout under consideration with parameters describing its behavior is on Fig.7.

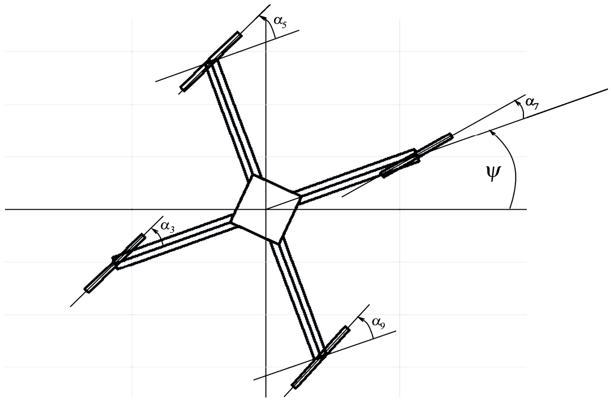

Fig. 7 The values indicative of instantaneous body and propellers position.

Physical model in system Dynast for mechatronic systems physical simulation is on Fig.8.

It can be see, that by means of rotational kinematic pairs are mutually interconnected quad-copter body model and propellers' model namely by force of translational velocity components (turning propellers centers move in the same way like their interface points on quad-copter body) and torques revolving with propellers operate between appropriate propeller and quad-copter body.

Actuating variable torque is simply controlled by force of PID regulator of quad-copter body position with its output limitation on level of . This regulator output is transferred on torque by means of zero order system (multiplication by constant).

The result (time dependencies) of simple simulation experiment is on Fig.9. Requested quad - copter body angular position jumps are:

$$\begin{array}{rl} 45^{\circ} & \left\{ \begin{array}{c} \text{pro } (t \geq 0.2s) \cap (t < 0.8s) \\ \text{pro } (t \geq 0.8s) \cap (t < 2s) \\ \text{pro } (t \geq 2s) \end{array} \right. \\ -30^{\circ} & \\ 20^{\circ} & \end{array}$$

(5)

Fig. 10 shows quad-copter body's position successive change at mentioned requested motion.

III. Conclusion

The paper is concerned with utilization of DH method coordinate systems introduction at kinematics and dynamism description of ideally rigid bodies system in space constrained by kinematic pairs and shows the universal physical models utilization possibility for dynamic behavior simulation of these systems.

Stated principle was used on quad-copter body motion model (yaw) levitating in existing altitude and its control in spin.

This contribution forms the first part of the works concerned with flying service robots (multi-copters) dynamics description.

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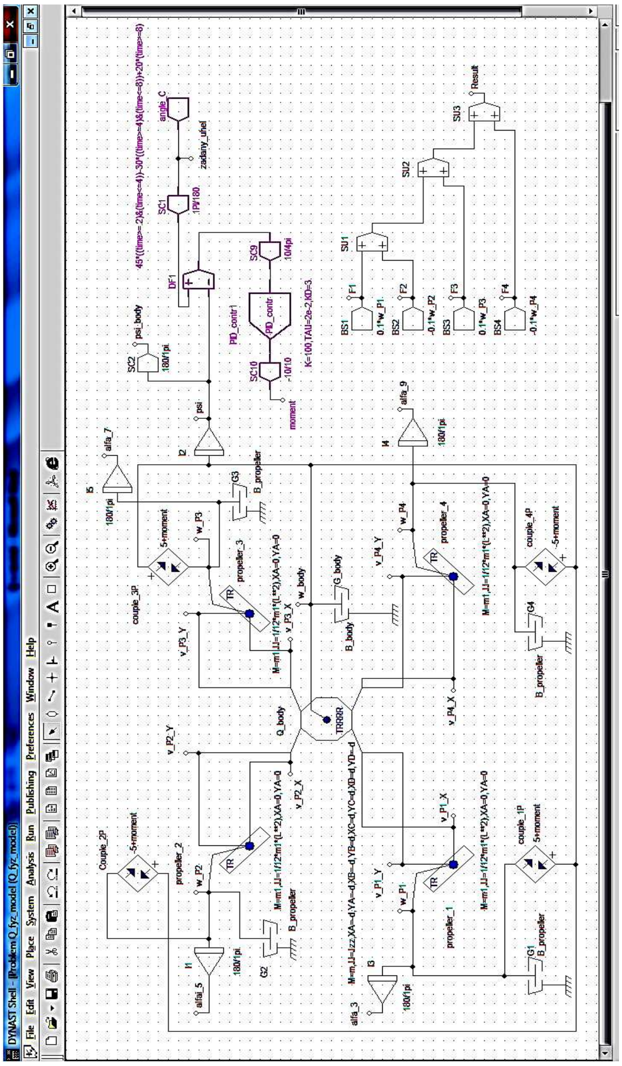

Fig. 8 Resulting physical model od quad–copter in plane.

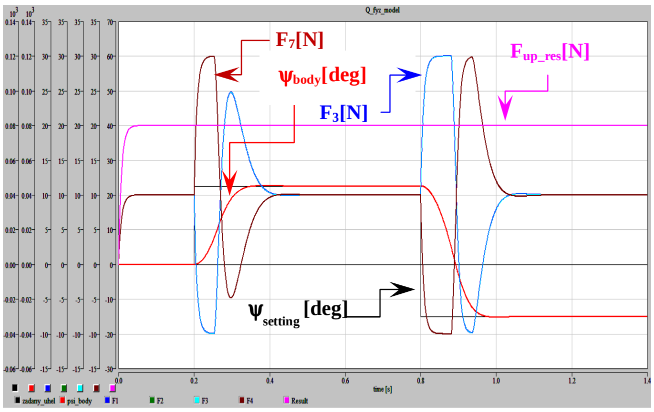

Fig. 9 Quad-copter levitation and its spin control in plane.

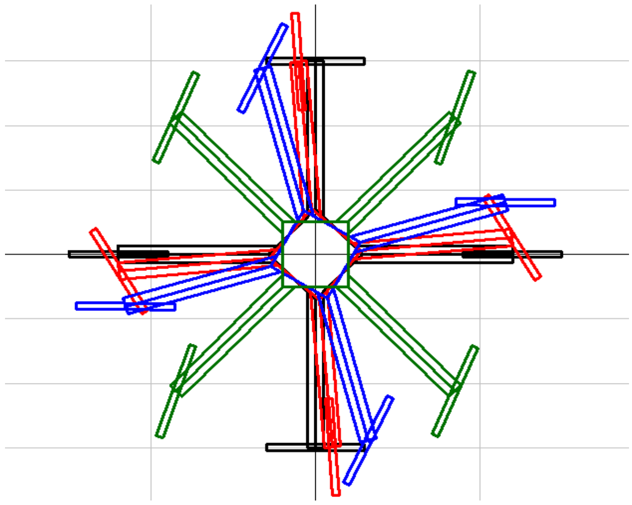

Fig. 10 Quad–copter model motion in plane

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