Discrete-Time State Estimation Using Unbiased FIR Filters with Minimized Variance

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Abstract: Optimal or unbiased estimators are widely used for state estimation and tracking. We propose a new minimum variance unbiased (MVU) finite impulse response (FIR) filter which minimizes the estimation error variance in the unbiased FIR (UFIR) filter. The relationship between the filter gains of the MVU FIR, UFIR and optimal FIR (OFIR) filters is found analytically. Simulations provided using a polynomial state-space model have shown that errors in the MVU FIR filter are intermediate between the UFIR and OFIR filters, and the MVU FIR filter exhibits better denoising effect than the UFIR estimates. It is also shown that the performance of MVU FIR filter strongly depends on the averaging interval of N points: by small N, the MVU FIR filter approaches UFIR filter and, if N is large, it becomes optimal.

Key–Words: State estimation, minimum variance, unbiased filter, FIR filter.

1 Introduction

The finite impulse response (FIR) estimators (smoothers, filters and predictors) have been under the development in the last three decades [1–7]. An important progress was achieved by Kwon, Kim and Park in [7], where the problem by combining the receding horizon strategy with the KF was solved. After that, the unbiased FIR (UFIR) filter was proposed for discrete-time system model in [8], and a fixed-lag FIR smoother was developed in [9] for continuous-time models. Quite recently, an UFIR filter was derived by Shmaliy in [10] and [11], for real-time state space models. Further, the p-shift optimal FIR (OFIR) estimator was obtained in [12] and [13] for time-invariant state space model. Using in part the results obtained in [13], Shmaily proposed in [14] a Kalman-like UFIR estimator for the time-variant case. In [15], a suboptimal FIR estimator was developed by using the extended KF strategy. Moreover, unified forms for KF and FIR filter and smoother were shown and investigated in [16]. Although the progress in FIR filtering opens new horizons in optimal and robust estimation of linear and nonlinear models, some well-recognized approaches such as the minimum variance unbiased (MVU) estimation still remain undeveloped in FIR filtering.

In this paper, a MVU FIR filter is derived for discrete time-variant state space model to minimize the variance in the UFIR filter proposed in [13]. Compared to the IIR filters, the MVU FIR filter inherits advantages of FIR structures and is more robust against the temporary modeling uncertainties. We use the following notations: $\mathbb{R}^n$ denotes the n dimensional Euclidean space, $E\{\cdot\}$ denotes the statistical averaging of the stochastic process or vector, $diag(e_1 \cdots e_m)$ represents a diagonal matrix with diagonal elements $e_1, \cdots, e_m$, $tr(M)$ is the trace of M, and I is the identity matrix of proper dimensions.

2 State-Space Model and Preliminaries

Motivated by the problems of state estimation and tracking often arising in signal processing and wireless systems, we consider a linear discrete-time system represented in state-space with the time-variant model

$$x_k = A_k x_{k-1} + B_k w_k \,, \tag{1}$$
$$y_k = C_k x_k + D_k v_k \,, \tag{2}$$

where $x_k \in \mathbb{R}^n$ is the state vector in Euclidean space, $y_k \in \mathbb{R}^p$ is the measurement vector, $A_k \in \mathbb{R}^{n\times n}$, $B_k \in \mathbb{R}^{n\times u}$, $C_k \in \mathbb{R}^{p\times n}$ and $D_k \in \mathbb{R}^{p\times v}$ are time-variant matrices, which are assumed to be known. The process noise $w_k \in \mathbb{R}^u$ and measurement noise

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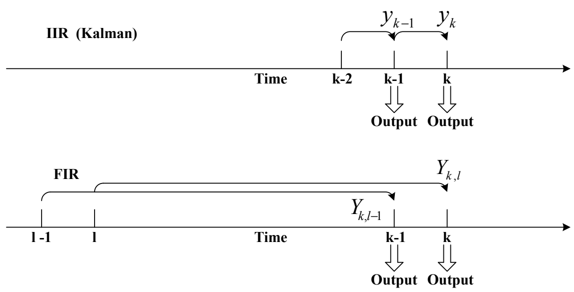

Figure 1: Operation time diagrams of the IIR and FIR structures.

$v_k \in \mathbb{R}^v$ are zero mean, $E\{w_k\} = 0$ and $E\{v_k\} = 0$, mutually uncorrelated and have arbitrary distributions and known covariances $Q(i,j) = E\{w_i w_j^T\}$, $R(i,j) = E\{v_i v_j^T\}$ for all i and j, meaning that w_k and v_k do not have to be white.

The FIR filter can be expressed as a linear combination of finite samples of measurements as

$$\hat{x}_{k|k} = K_k Y_{k,l}, \tag{3}$$

where $l = k - N + 1$ is the starting point of the horizon, N is the horizon length, $\hat{x}_{k|k}$ is the estimate[1], $Y_{k,l}$ is a vector measurements collecting on a horizon $[l,k]$, and K_k is the filter gain determined by a given performance criterion.

The operation principles of the FIR and IIR filters are illustrated in Fig.1. A distinct difference is that only one most recent measurement is used in IIR (Kalman) filtering to provides the estimate, while FIR estimators employ N most recent measurements. This leads to N times larger computation time than in IIR filtering. However, some good properties such as the BIBO stability and better robustness are guaranteed at the cost of extra computation time. We formulate the problem as follows: Given the model, (1) and (2), we would like to derive a MVU FIR filter minimizing the variance in the UFIR filter, by

$$K_k = \arg\min_{K_k} E\left\{\left(x_k - \hat{x}_{k|k}\right)\left(x_k - \hat{x}_{k|k}\right)^T\right\}. \tag{4}$$

We also wish to compare errors in three different FIR filters (MVU FIR derived in this paper and the UFIR and OFIR filters proposed in [14] and [17] respectively) to each other, and analyze the trade-off based on a polynomial state-space model.

[1] $\hat{x}_{k|k}$ means the estimate at k utilizing on the measurements from the past to k.

3 MVU FIR Filter

In order to derive the MVU FIR filter on a horizon of N past measurements from l to k, we represent (1) and (2) in a batch form following [14] and [17] as

$$X_{k,l} = A_{k,l} x_l + B_{k,l} W_{k,l}, \tag{5}$$

$$Y_{k,l} = C_{k,l} x_l + H_{k,l} W_{k,l} + D_{k,l} V_{k,l}. \tag{6}$$

Here, $X_{k,l} \in \mathbb{R}^{Nn}$, $Y_{k,l} \in \mathbb{R}^{Np}$, $W_{k,l} \in \mathbb{R}^{Nu}$ and $V_{k,l} \in \mathbb{R}^{Nv}$ are specified as, respectively,

$$X_{k,l} = \left[x_k^T x_{k-1}^T \cdots x_l^T\right]^T, \tag{7}$$

$$Y_{k,l} = \left[y_k^T y_{k-1}^T \cdots y_l^T\right]^T, \tag{8}$$

$$W_{k,l} = \left[w_k^T w_{k-1}^T \cdots w_l^T\right]^T, \tag{9}$$

$$V_{k,l} = \left[v_k^T v_{k-1}^T \cdots v_l^T\right]^T. \tag{10}$$

The extended model matrix $A_{k,l} \in \mathbb{R}^{Nn\times n}$, process noise matrix $B_{k,l} \in \mathbb{R}^{Nn\times Nu}$, observation matrix $C_{k,l} \in \mathbb{R}^{Np\times n}$, auxiliary process noise matrix $H_{k,l} \in \mathbb{R}^{Np\times Nu}$ and measurement noise matrix $D_{k,l} \in \mathbb{R}^{Np\times Nv}$ are all time-variant and dependent on the current time k and the horizon length N. Model (1) and (2) suggests that these matrices can be written as, respectively

$$A_{k,l} = [\, \mathcal{A}_{k,l+1}^T \, \mathcal{A}_{k-1,l+1}^T \, \cdots \, \mathcal{A}_{l+1,l+1}^T \; I\,]^T, \tag{11}$$

$$C_{k,l} = \bar{C}_{k,l} A_{k,l}, \tag{13}$$

$$H_{k,l} = \bar{C}_{k,l} B_{k,l}, \tag{14}$$

$$D_{k,l} = \mathrm{diag}\left(D_k D_{k-1} \cdots D_l\right), \tag{15}$$

with

$$\mathcal{A}_{\psi,\zeta} = \begin{cases} A_\psi A_{\psi-1} \cdots A_\zeta, & \text{if } \psi > \zeta \\ A_\psi, & \text{if } \psi = \zeta \end{cases}, \tag{16}$$

$$\bar{C}_{k,l} = \mathrm{diag}\left(C_k C_{k-1} \cdots C_l\right), \tag{17}$$

where $\psi \geqslant \zeta$. Note that the state equation specified by (5) and (6) at the initial point l is $x_l = x_l + B_l w_l$, suggesting that w_l is zero-valued. That is, the initial state x_l is required to be known or estimated optimally. The following lemma will be used to derive the MVU FIR filter.

Lemma 1 *The trace optimization problem is given by:*

$$\arg\min_{K} \mathrm{tr}[(KA-B)C(KA-B)^T + (KG-F)D(KG-F)^T + KPK^T], \tag{18}$$

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$$B_{k,l} = \begin{bmatrix} B_k & \mathcal{A}_{k,k}B_{k-1} & \cdots & \mathcal{A}_{k,l+2}B_{l+1} & \mathcal{A}_{k,l+1}B_l \\ 0 & B_{k-1} & \cdots & \mathcal{A}_{k-1,l+2}B_{l+1} & \mathcal{A}_{k-1,l+1}B_l \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & B_{l+1} & \mathcal{A}_{l+1,l+1}B_l \\ 0 & 0 & \cdots & 0 & B_l \end{bmatrix}, \quad (12)$$

subject to $KE = H$, where $C = C^T > 0$, $D = D^T > 0$, $P = P^T > 0$, and A, B, C, D, E, F, G, H, and P are constant matrices of proper dimensions. The solution to this optimization problem is as follows:

$$K = [\,H \;\; B \;\; F\,] \times \begin{bmatrix} \left(E^T\Xi^{-1}E\right)^{-1}E^T\Xi^{-1} \\ CA^T\Xi^{-1}\left(I - E\left(E^T\Xi^{-1}E\right)^{-1}E^T\Xi^{-1}\right) \\ DG^T\Xi^{-1}\left(I - E\left(E^T\Xi^{-1}E\right)^{-1}E^T\Xi^{-1}\right) \end{bmatrix}, \quad (19)$$

where $\Xi = ACA^T + GDG^T + P$.

The proof can be obtained following similar lines provided in [18] and is omitted here.

3.1 Design of MVU FIR Filter

In order to minimize the variance in UFIR filter, we decompose K_k as

$$K_k = \bar{K}_k + \tilde{K}_k, \quad (20)$$

where $\tilde{K}_k$ is an unknown additive term to be determined, and $\bar{K}_k$ is the known unbiased gain [14] specified by

$$\bar{K}_k = \mathcal{A}_{k,l+1}(C_{k,l}^T C_{k,l})^{-1}C_{k,l}^T \quad (21)$$

and derived to satisfy the unbiasedness condition $E\{x_k\} = E\{\hat{x}_{k|k}\}$. From (5), the system state x_k can be constructed as

$$x_k = \mathcal{A}_{k,l+1}x_l + \bar{B}_{k,l}W_{k,l}, \quad (22)$$

where $\bar{B}_{k,l}$ is the first row in (12). Substituting x_k with (22), $\hat{x}_{k|k}$ with (3) using (20) in the unbiasedness condition yields the constraint

$$\tilde{K}_k C_{k,l} = 0, \quad (23)$$

where the fact known from [14] that $\bar{K}_k C_{k,l} = \mathcal{A}_{k,l+1}$ is used. Now, our objective is to obtain $\tilde{K}_k$ in a way such that the estimate has the minimum variance

$$\tilde{K}_k = \arg\min_{\tilde{K}_k} E\left\{\mathrm{tr}\left[\left(x_k - \hat{x}_{k|k}\right)\left(x_k - \hat{x}_{k|k}\right)^T\right]\right\} \quad (24)$$

with constraint $\tilde{K}_k C_{k,l} = 0$. Substituting (4) and (3) with consideration of (20) into (24) yields

$$\begin{aligned}\tilde{K}_k = \arg\min_{\tilde{K}_k} E\{tr\,[(&A_{k,l+1}x_l + \bar{B}_{k,l}W_{k,l} \\ &- \left(\bar{K}_k + \tilde{K}_k\right)C_{k,l}x_l - \left(\bar{K}_k + \tilde{K}_k\right) \\ &\times (H_{k,l}W_{k,l} + D_{k,l}V_{k,l}))(\cdots)^T]\},\end{aligned} \quad (25)$$

where $(\cdots)$ denotes the same term as its previous term. By taking into account of $\bar{K}_k C_{k,l} = \mathcal{A}_{k,l+1}$, observing that systems noise vector $W_{k,l}$ and measurement noise $V_{k,l}$ are pairwise independent, providing the averaging, and rearranging the terms, (25) becomes

$$\begin{aligned}\tilde{K}_k = \arg\min_{\tilde{K}_k} \mathrm{tr}\Big[&\left(\tilde{K}_k H_{k,l} + \bar{K}_k H_{k,l} - \bar{B}_{k,l}\right) \\ &\times\Theta_{w,l}(\dots)^T + \tilde{K}_k\Delta_{x,l}\tilde{K}_k^T \\ &+(\tilde{K}_k + \bar{K}_k)\Delta_{v,l}(\dots)^T\Big],\end{aligned} \quad (26)$$

where the auxiliary matrices are

$$\Theta_{w,l} = E\left\{W_{k,l}W_{k,l}^T\right\}, \quad (27)$$

$$\Delta_{x,l} = C_{k,l}E\left\{x_l x_l^T\right\}C_{k,l}^T, \quad (28)$$

$$\Delta_{v,l} = D_{k,l}E\left\{V_{k,l}V_{k,l}^T\right\}D_{k,l}^T. \quad (29)$$

Next, by using the result of Lemma 1 with the replacements $A \leftarrow H_{k,l}$, $B \leftarrow (\bar{B}_{k,l} - \bar{K}_k H_{k,l})$, $C \leftarrow \Theta_{w,l}$, $D \leftarrow \Delta_{v,l}$, $E \leftarrow C_{k,l}$, $F \leftarrow -\bar{K}_k$, $G \leftarrow I$, $H \leftarrow 0$, and $P \leftarrow \Delta_{x,l}$, the solution to the optimization problem (26) can be obtained as

$$\tilde{K}_{k,l} = \Omega_{k,l}(I - \Lambda_{k,l}), \quad (30)$$

where

$$\Omega_{k,l} = (\bar{B}_{k,l}\Theta_{w,l}H_{k,l}^T - \bar{K}_k\Delta_{w+v,l})\Delta_{x+w+v,l}^{-1}, \quad (31)$$

$$\Delta_{w,l} = H_{k,l}\Theta_{w,l}H_{k,l}^T, \quad (32)$$

$$\Delta_{w+v,l} = \Delta_{w,l} + \Delta_{v,l}, \quad (33)$$

$$\Delta_{x+w+v,l} = \Delta_{x,l} + \Delta_{w,l} + \Delta_{v,l}, \quad (34)$$

$$\Lambda_{k,l} = C_{k,l}(C_{k,l}^T\Delta_{x+w+v,l}^{-1}C_{k,l})^{-1}C_{k,l}^T\Delta_{x+w+v,l}^{-1}. \quad (35)$$

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Table 1: MVU FIR Filtering Algorithm

Stage	
Given:	$N \geq n, \ l = k - N + 1$
Solve:	$\Delta_{x,l}$ using the DARE (36)
Find:	$\bar{K}_k$ by (21), $\Omega_{k,l}$ by (31), and $\Lambda_{k,l}$ by (35)
Compute:	$\hat{x}_{k\|k} = [\bar{K}_k + \Omega_{k,l}(I - \Lambda_{k,l})]Y_{k,l}$

In order to compute (30), the variance of the initial state $\Delta_{x,l}$ is required. Using the approximation $E\{x_l x_l^T\} \approx \hat{x}_{l|k}\hat{x}_{l|k}^T$, the following discrete algebraic Riccati equation (DARE) can be used to find $\Delta_{x,l}$, as in [13],

$$Y_{k,l}Y_{k,l}^T\Delta_{w+v,l}^{-1}\Delta_{x,l} - \Delta_{x,l}\Delta_{w+v,l}^{-1}\Delta_{x,l} \\ -2\Delta_{x,l} - \Delta_{w+v,l} = 0\,. \tag{36}$$

The MVU FIR filter is now specified by the following theorem.

Theorem 2 *Given the discrete time-variant state space model (1) and (2) with zero mean and mutually independent noise vectors w_k and v_k having arbitrary distributions and known covariances, the MVU FIR filter utilizing measurements from l to k can be written as*

$$\hat{x}_{k|k} = \left[\bar{K}_k + \Omega_{k,l}(I - \Lambda_{k,l})\right] Y_{k,l}, \tag{37}$$

where $Y_{k,l} \in \mathbb{R}^{Np}$ is the measurement vector given by (8), $\bar{K}_k$ is obtained by (21) with $\mathcal{A}_{k,l+1}$ and $C_{k,l}$ specified by (16) and (13) respectively, $\Omega_{k,l}$ and $\Lambda_{k,l}$ are determined using (31) and (35) by solving the DARE (36).

The proof has been provided by (20)-(36).

Note that the horizon length N should be chosen such that the inverse in $\bar{K}_k$ specified by (21) exists. In general, N can be set as $N \geq n$, where n is the number of model state. Table I summarizes the steps in the MVU FIR estimation algorithm, in which $\Delta_{w,v}$ and $\Delta_{v,l}$ are assumed to be known for measurements available from l to k. Given N, solve the DARE (36). Then, compute $\bar{K}_k$ according to (21) by using (11) and (13). With (31) and (35), the MVU FIR estimate $\hat{x}_{k|k}$ can be obtained by (37) at time k.

3.2 FIR Filter Gains

As can been seen from Theorem 1, the term $I - \Lambda_{k,l}$ in $\tilde{K}_k$ is introduced by the constraint $\tilde{K}_k C_{k,l} = 0$ evolved from the unbiasedness condition. Specifically, the existence of term $I - \Lambda_{k,l}$ ensures the unbiasedness of the FIR filter. If we neglect this constraint, the matrix E in Lemma 1 will be replaced by the zero matrix when solving the optimization problem (26), thus leading to $\Lambda_{k,l} = 0$. At this point, we have $\tilde{K}_k = \Omega_{k,l}$ and the MVU FIR filter gain K_k becomes the OFIR filter gain $\hat{K}_k$, due to the following equality:

$$\hat{K}_k = \bar{K}_k + \Omega_{k,l}\,, \tag{38}$$

with

$$\hat{K}_k = (\bar{K}_k\Delta_{x,l} + \bar{B}_{k,l}\Theta_{w,l}H_{k,l}^T)\Delta_{x+w+v,l}^{-1}\,. \tag{39}$$

which is given in [17] as the filter gain of the OFIR filter. On the other hand, it follows from (20) and (30) that

$$K_k = \hat{K}_k - \Omega_{k,l}\Lambda_{k,l} = \bar{K}_k + \Omega_{k,l}(I - \Lambda_{k,l}). \tag{40}$$

This equality provides the analytical relationships between the MVU FIR filter gain K_k, the OFIR filter gain $\hat{K}_k$, and the UFIR filter gain $\bar{K}_k$. Note that the structure of the MVU FIR filter is shown to be consistent with the UFIR and OFIR filters, suggesting that the UFIR, MVU FIR and OFIR filters do not get away essentially from each other and all the FIR filters can be transformed to each other by adding or subtracting the corresponding terms. Moreover, the UFIR filter gain $\bar{K}$ plays a fundamental role in the FIR filter design being independent on the noise statistics

4 Examples and Applications

Extensive comparisons between FIR filters with Kalman methods can be found in [13, 14, 17, 19–21]. We therefore mostly compare the MVU FIR filter with the UFIR and OFIR filters in order to investigate the trade-off. Towards this end, a two-state polynomial state space models (1) and (2), specified with $B_k = [1, 1]^T$, $D_k = 1$, $C_k = [1, 0]$, and

$$A_k = \begin{bmatrix} 1 & (1 + d_k)\tau \\ 0 & 1 \end{bmatrix} \tag{41}$$

is employed, where τ is a constant in unit of time, and d_k varies with time. Note that this kind of systems is commonly used to describe the “velocity jumps” in moving target tracking and the “frequency jumps” in oscillators.

In the first simulation, all the methods were applied in an ideal environment. That is, all the parameters of system model, including the variances of noises, are known completely in the entire estimation

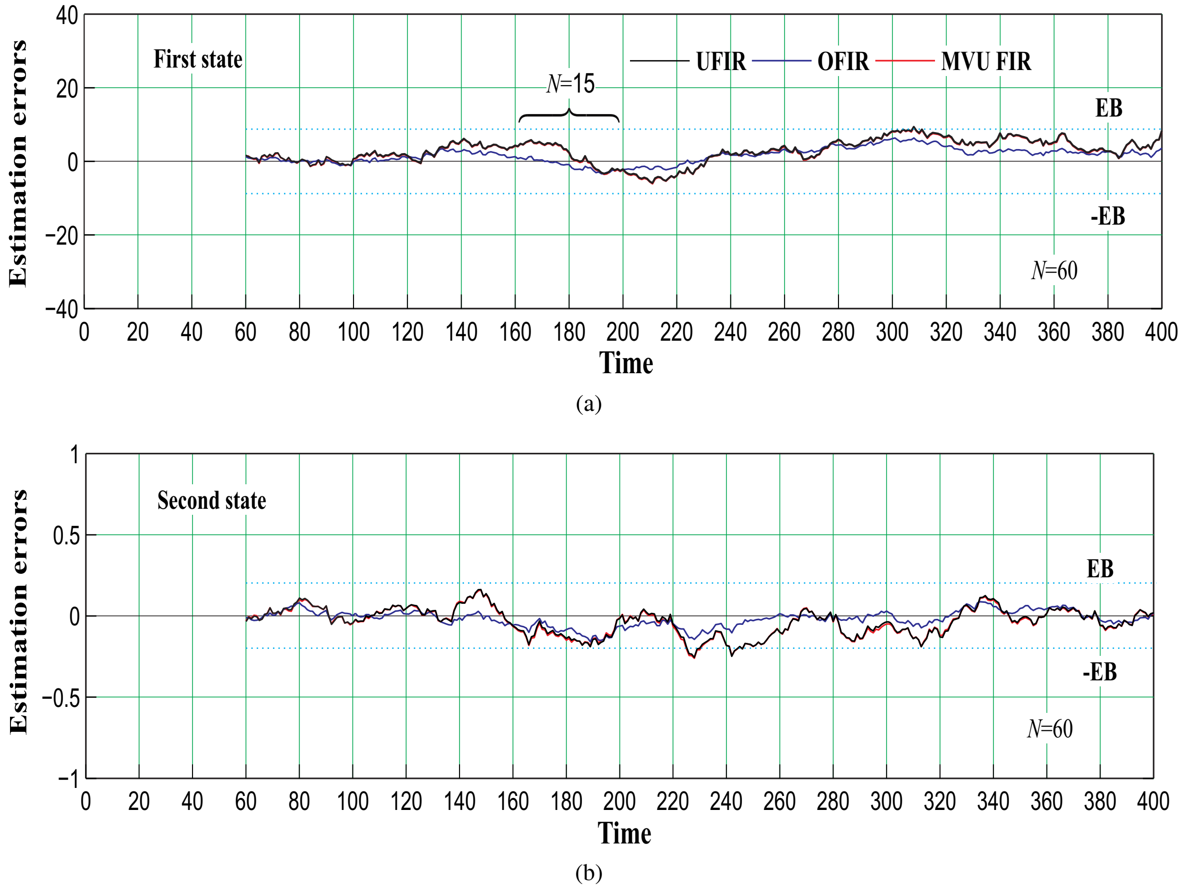

Figure 2: Estimation errors for an accurate model: (a) the first state and (b) the second state.

process. The model parameter is set to be $d_k = 20$ if $160 \leq k \leq 200$ and $d_k = 0$ otherwise. The variances of process noise w_k and measurement noise v_k are $\sigma_w^2 = 10^{-4}$ and $\sigma_v^2 = 10^2$, respectively. Since the system model is time-variant, two different optimum horizons were found following [21] to be $N_{opt} = 60$ and $N_{opt} = 15$ for the models with $d_k = 0$ and $d_k = 20$, respectively. For the first and second states at time k, the estimation error bounds (EBs) have been respectively calculated as in [14] and employed as measures of estimation accuracy. The process was simulated over 400 subsequent points.

The estimation errors are given in Fig. 2. With respect to the first state, errors in OFIR filter is smaller than in the MVU FIR and UFIR filters as expected. Note that the OFIR filter is obtained by minimizing the mean square errors (MSEs), thus leading to the most accurate estimates. On the other hand, UFIR and OFIR perform very close to each other and the difference between them is indistinguishable in Fig. 2. This is mainly due to the optimal horizon used, which reduces the variance of estimation error in UFIR automatically. For the second state, a similar situation is observed; however, FIR methods are less sensitive to the value of N in this case. It is also seen that all the FIR filters trace well within a gap between EB and -EB formed with $N = 60$.

To show the difference between the errors more clearly, we conducted another experiment with $\sigma_w^2 = 10^{-3}$ and $\sigma_v^2 = 10$ using a time-invariant model with $d_k = 0$. For polynomial model, corresponding N_{opt} for different states can be determined separately, the root squares of MSEs of the first and second states are respectively shown in Fig. 3a and Fig. 3b. A analysis of Fig. 3 leads to several critical inferences:

- The MSEs are concave on N with polynomial model. When $N < N_{opt}$, the denoising effect in all the FIR filters becomes better with the increase of N. All the FIR filters offer best estimates at their corresponding optimal horizons. On the other hand, an increase in N results in an increase in the estimation bias that can easily be seen in Fig.3 with $N > N_{opt}$.

- The errors in MVU FIR filter are intermediate between the UFIR and OFIR filters. Specifically, the MVU FIR filter performs better than the UFIR filter but worse than the optimal one.

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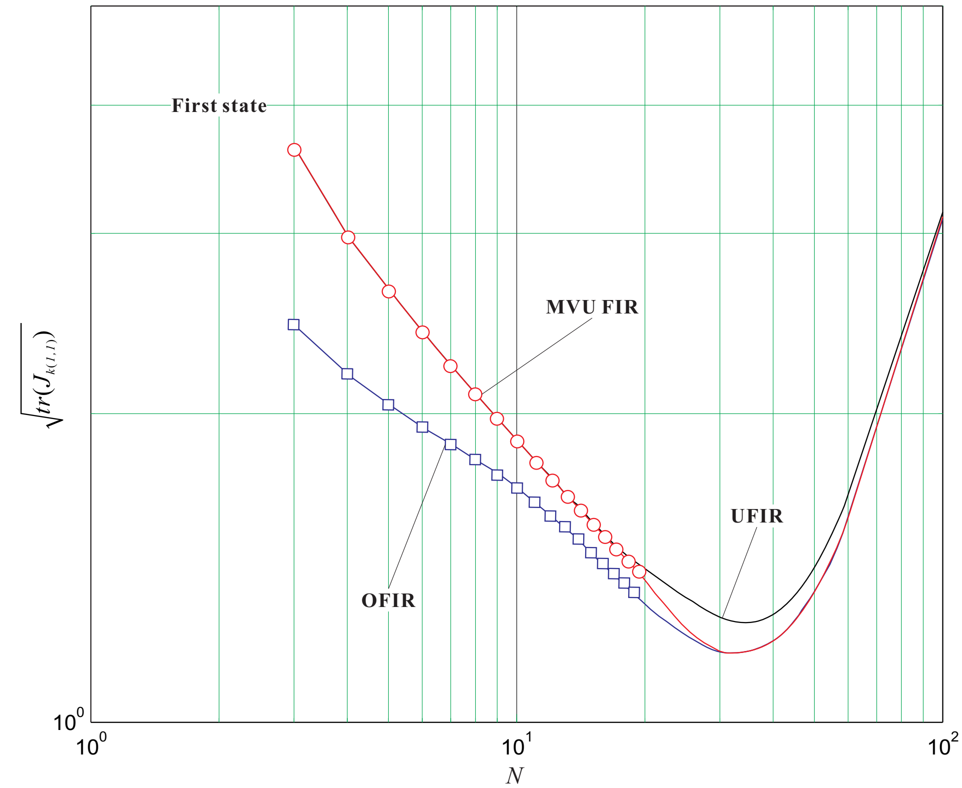
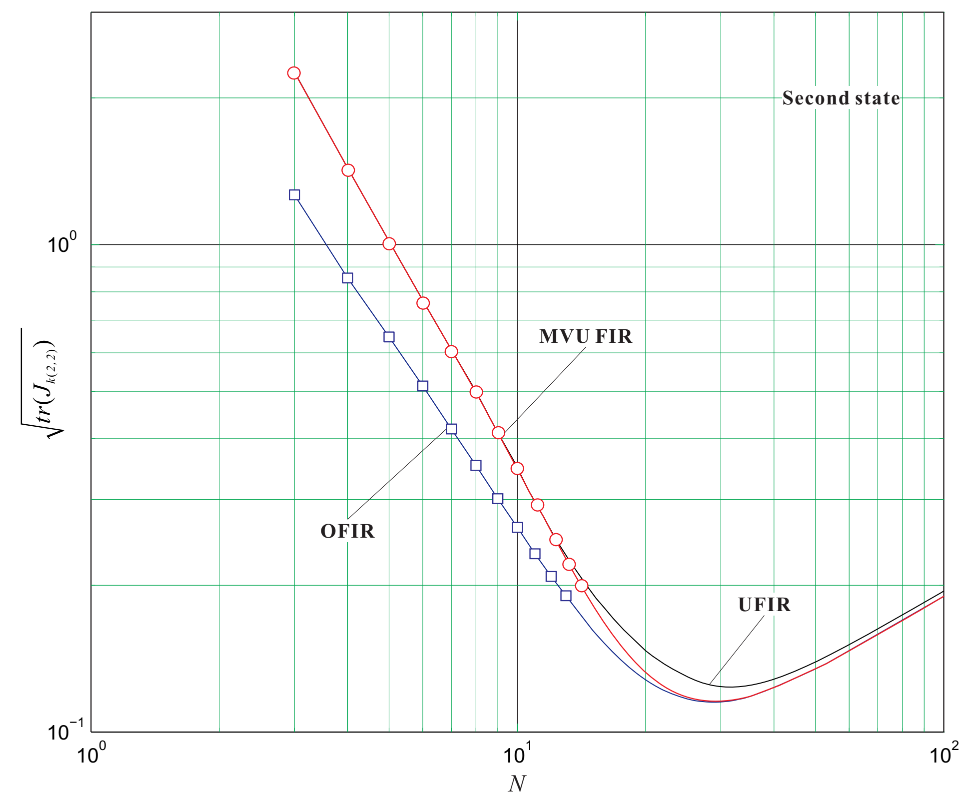


(a)



(b)

Figure 3: Root square of MSEs as functions of N with $\sigma_w^2 = 10^{-3}$ and $\sigma_v^2 = 10$: (a) the first state and (b) the second state.

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Another critical peculiarity that still has not been shown in the literature is that the UFIR filter is also highly successful in denoising, and it tracks closely to the MVU FIR filter (the optimized UFIR one), when N is small. Overall, the MVU FIR filter behaves as the UFIR filter with small N and as the OFIR filter with large N.

- By increasing the horizon N, all FIR estimates converge to the OFIR estimate. In comparison with the UFIR filter, the rate of convergence of the MVU FIR filter is much faster, thus leading to smaller optimal horizon.

Note that noise in the UFIR filter is reduced only by the averaging provided using the estimation horizon N. In the MVU FIR filter, however, additional optimization operation is introduced to enhance the effect of noise reduction, even making the MVU FIR method achieve similar accuracy with the OFIR filter at the same optimal horizon point.

5 Conclusions

We have derived a new form of the MVU FIR filter by minimizing the variance in the UFIR filter which completely ignores the noise statristics. As was expected, the MVU FIR filter has reduced the random amount of errors and shown better ability of denoising. Its estimates converge to the OFIR one by increasing the averaging horizon. The structure of the MVU FIR filter was shown to be consistent to the OFIR filter and the MSEs intermediate between the UFIR and OFIR filters.

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