Graphical analysis of robust stability for fractional order time-delay systems and integer order PID controllers

Radek Matušů and Roman Prokop

Abstract—The main goal of this contribution is to present the universal graphical tool for investigation of robust stability and especially its possible application to analysis of feedback control loops which include a fractional order time-delay controlled systems with parametric uncertainty and fixed integer order PID controllers. The robust stability testing is based on plotting the value sets for closed-loop characteristic quasi-polynomial and subsequent application of the zero exclusion condition. The effectiveness but also easy utilization are demonstrated by the set of computational examples for the case of uncertain gain, uncertain time constant and uncertain time-delay term, respectively.

Keywords—Fractional Order Control, PID Controllers, Robust Stability Analysis, Time-Delay Systems, Zero Exclusion Condition.

I. INTRODUCTION

The fractional order calculus represents more than 300-year-old branch of mathematics that is focused on differentiation and integration under an arbitrary, real or even complex, order of the operation [1] – [7]. Recently, the fractional order has found the real-life application possibilities in many areas such as bioengineering, viscoelasticity, electronics, robotics, control theory or signal processing [8]. Especially in the field of control engineering, it seems that the true fractional order “boom” has exploded lately as many new research works have appeared.

The mathematical model of the controlled system practically never exactly matches its real behaviour. This fact is typically caused by the effort to construct a simple-to-use linear model in which the more complex properties such as nonlinearities, time-variant behaviour or very fast dynamics are neglected. Moreover, the physical parameters of the system can change due to various reasons. All these factors can be taken into consideration by using the uncertain model instead of the ordinary fixed one. The very popular approach to uncertainty modelling is based on models with fixed structure (order), but not exactly known parameters, which are supposed to lie within given bounds. Such models are called systems with parametric uncertainty and the frequent task is to analyze their robust stability, i.e. if the stability is ensured for all possible values of uncertain parameters. Obviously, several researchers have already combined the issue of robust stability of systems under parametric uncertainty with fractional order systems – e.g. [9] – [14].

This contribution deals with a graphical approach to robust stability investigation for feedback control loops with fractional order time-delay plants and integer order PID controllers. The testing is based on plotting the value sets of a closed-loop characteristic quasi-polynomial and subsequent application of the zero exclusion condition [15]. The computational examples present the analyses for the cases of uncertain gain, uncertain time constant and uncertain time-delay term and they show both robustly stable and unstable events.

II. FUNDAMENTALS OF FRACTIONAL ORDER SYSTEM DESCRIPTION

The fractional order calculus is based on generalization of differentiation and integration to an arbitrary order. This generalization has resulted in the introduction of basic continuous differintegral operator [1], [2], [4], [8]:

\[
^{\alpha}D_t^\alpha \left\{ \begin{array}{ll}
\frac{d^\alpha}{dt^\alpha} & \text{Re} \alpha > 0 \\
1 & \text{Re} \alpha = 0 \\
\int (d\tau)^{-\alpha} & \text{Re} \alpha < 0
\end{array} \right.
\] (1)

where \( \alpha \) is the order of the differintegration (typically \( \alpha \in \mathbb{R} \)) and \( a \) is a constant connected with initial conditions. The differintegral can be defined in various ways. The tree most common are Riemann-Liouville, Grünwald-Letnikov and Caputo definitions.

The Laplace transform of the differintegral is given by [4], [16]:

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L\{e^{-\alpha t}D^n_{\omega}f(t)\} = \int_0^\infty e^{-\alpha t}D^n_{\omega}f(t)dt =
\begin{equation}
\left. s^nF(s) - \sum_{n=0}^{\infty} s^{n-1}D^n_{\omega-m-1}f(t) \right|_{t=0}
\end{equation}

where integer \( n \) lies within \((-1 < \alpha \leq n)\).

The (time-delay free) fractional order transfer function can be written as [3], [5]:
\begin{equation}
G(s) = \frac{B(s^\beta)}{A(s^\alpha)} = \frac{b_0s^\beta + b_{\alpha}s^{\beta-\alpha} + \cdots + b_{\beta}s^0}{a_0s^{\alpha} + a_{\alpha}s^{\alpha-\beta} + \cdots + a_{\beta}s^0}
\end{equation}

where \( a_k \) with \((k = 0, \ldots, n)\) and \( b_k \) with \((k = 0, \ldots, m)\) denote constants, and \( \alpha_k \) with \((k = 0, \ldots, n)\) and \( \beta_k \) with \((k = 0, \ldots, m)\) are arbitrary real numbers. According to [4], [5], one can assume inequalities \( \alpha_k > \alpha_{k+1} > \cdots > \alpha_0 \) and \( \beta_k > \beta_{k+1} > \cdots > \beta_0 \) without loss of generality. In this paper, the controlled time-delay system is supposed generally as:
\begin{equation}
G(s) = \frac{B(s^\beta)}{A(s^\alpha)} e^{-\Theta s}
\end{equation}

III. ANALYSIS OF ROBUST STABILITY FOR SYSTEMS WITH PARAMETRIC UNCERTAINTY

The robust stability of the feedback control system will be investigated by means of its closed-loop characteristic polynomial (actually, a quasi-polynomial in the case of this contribution).

The fractional order version of the continuous-time uncertain polynomial with vector of uncertainty \( q \) and coefficient functions \( \rho_s \) can be written as:
\begin{equation}
p(s,q) = \rho_0(q)s^{\alpha_0} + \rho_{\alpha_{k+1}}(q)s^{\alpha_{k+1}} + \rho_1(q)s^{\alpha_1} + \rho_{\alpha_m}(q)s^{\alpha_m}
\end{equation}

Then, the polynomial family is defined by [15]:
\begin{equation}
P = \{p(s,q) : q \in Q\}
\end{equation}

where \( Q \) is the uncertainty bounding set restricting the uncertain parameters. Commonly, \( Q \) is supposed as a multidimensional box, i.e. individual parameters are bounded by intervals.

The polynomial family (6) is robustly stable if and only if \( p(s,q) \) is stable for all \( q \in Q \). The selection of specific tool for investigation of robust stability depends mainly on the structure of uncertainty. Generally, the higher level of relation among coefficients means more complex robust stability analysis and brings necessity of more sophisticated techniques. Nevertheless, a graphical method based on combination of the value set concept and the zero exclusion condition [15] is unique from the viewpoint of its universal applicability. It can be applied for wide range of uncertainty structures and it is usable also for various regions of stability (so called robust D-stability). The detailed information on robust stability analysis under parametric uncertainty can be found in [15] and subsequently e.g. in [17], [18]. Finally, the works [9] – [12] extended the idea of the value set concept also to fractional order uncertain polynomials.

According to [15], the value set at given frequency \( \omega \in \mathbb{R} \) is:
\begin{equation}
p(j\omega, Q) = \{p(j\omega,q) : q \in Q\}
\end{equation}

Practical creation of the value sets can be done by substituting \( s \) for \( \omega \in \mathbb{R} \), fixing \( \omega \in \mathbb{R} \) and letting \( q \) range over \( Q \).

The zero exclusion condition for Hurwitz stability of family of continuous-time polynomials (6) is defined [15]: Suppose invariant degree of polynomials in the family, pathwise connected uncertainty bounding set \( Q \), continuous coefficient functions \( \rho_s(q) \) for \( k = 0, 1, 2, \ldots, n \) and at least one stable member \( p(s,q^0) \). Then the family \( P \) is robustly stable if and only if:
\begin{equation}
0 \notin p(j\omega, Q) \quad \forall \omega \geq 0
\end{equation}

Authors of the papers [9], [11], [12] construct the value sets for the fractional order families of polynomials mainly on the basis of fact that the fractional power of \( j\omega \) can be written as:
\begin{equation}
(j\omega)^\alpha = \omega^{\frac{\alpha}{2}}\left(\cos\frac{\pi}{2} + j\sin\frac{\pi}{2}\right)
\end{equation}

and on the subsequent analysis of vertices and exposed edges.

Within this contribution, the value sets are plotted for closed-loop characteristic quasi-polynomials of the feedback control loop with the uncertain time-delay fractional order plant and fixed integer order PID controller. Their visualization is based on sampling the uncertain parameters and on calculation of partial points of the value sets for an assumed frequency range. Thanks to the applied sampling (brute-force) method, the value sets of quasi-polynomials can be easily computed and consequently the robust stability can be analyzed with the assistance of standard zero exclusion condition.

IV. COMPUTATIONAL EXAMPLES

Suppose a fractional order time-delay plant given by transfer function:
\begin{equation}
G(s,k,T,\Theta) = \frac{K}{T^\alpha s^\alpha + 1} e^{-\Theta s}
\end{equation}

where \( K \) represents a gain, \( T \) is a time-constant, and \( \Theta \) stands for a time-delay term. Nominal values of the parameters define the specific system:

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\[ G_p(s) = \frac{5}{10s^{0.9} + 1}e^{-10s} \]  

(11)

However, the parameters of really controlled systems are assumed to be uncertain, i.e. they can lie within given intervals. Their specific values will be stated successively in (15)-(18).

The PID controller for nominal plant (11) was obtained by using the FOMCON Toolbox for Matlab [19], [20] and its routine “iopid_tune”. More specifically, as shown in Fig. 1, the Oustaloup filter based [21] approximation resulted in the integer order model:

\[ G_i(s) = \frac{4.89523}{14.6777s + 1}e^{-3.86545s} \]  

(12)

which was then utilized in standard Cohen-Coon method for PID controller design. The obtained compensator is:

\[ C(s) = K_p + \frac{K_i}{s} + K_d s = 0.506221 + \frac{0.0278868}{s} + 1.48957s \]  

(13)

The graphical comparison of step responses of the original fractional-order nominal model (11) and its integer-order approximation (12) obtained with the assistance of the FOMCON Toolbox is shown in Fig. 2.

Fig. 1 GUI “iopid_tune” from the FOMCON Toolbox [19]

The main object of interest, robust stability of the closed-loop control system, will be tested through the family of characteristic quasi-polynomials:

\[ p_{cl}(s,K,T,\Theta) = \left( Ts^{0.9} + 1 \right)s + Ke^{-\Theta s}(K_d s^2 + K_r s + K_i) \]  

(14)

where one of the plant (10) parameters \( K, T, \Theta \) can vary according to (15)-(18) and where \( K_p, K_i \) and \( K_d \) are fixed PID controller parameters taken from (13).

First, only the gain is supposed to lie within the interval while time constant and time-delay term remain fixed, i.e.:

\[ K = \{4.5, 5.5\}; \quad T = 10; \quad \Theta = 10 \]  

(15)

Fig. 3 Value sets for controller (13) and plant with (15)
The value sets for the corresponding family of closed-loop characteristic quasi-polynomials consist of straight lines. The Fig. 3 shows these value sets for the range of frequencies from 0 to 3 with the step 0.005. At each frequency, $K$ is sampled within given interval with the step 0.01 (that means each line consists of 101 points). Then, the better view of the situation near the origin of the complex plane is provided by closer look in Fig. 4. Obviously, the zero point is not included in the value sets. Consequently, because the family contains at least one stable member and the zero is excluded, the family (14) with parameters (15) is robustly stable, so the closed-loop control system is robustly stable.

Now, the gain is the only uncertain parameter again, but the assumed bounds are a bit wider:

$$K = \langle 4, 6 \rangle; \quad T = 10; \quad \Theta = 10$$  \hfill (16)

The Fig. 5 depicts the value sets for the new interval under the same conditions as in the previous case. The zoomed version is shown in Fig. 6. As can be seen, the origin of the complex plane is included in the value sets and thus the feedback loop with plant parameters (16) is robustly unstable.

Next, the time constant is the uncertain parameter while the gain and time-delay term are fixed:

$$K = 5; \quad T = \langle 9, 11 \rangle; \quad \Theta = 10$$  \hfill (17)

The relevant value sets can be seen in Fig. 7 and the zoomed view in Fig. 8. The frequency range is $\omega = 0.005:3$ and the time constant is sampled $T = 9:0.05:11$. The complex plane origin is excluded from the value sets, the family contains at least one stable member and thus the family is robustly stable.
Finally, time-delay term is considered as the uncertain one:

\[ K = 5; \quad T = 10; \quad \Theta = \langle 9, 11 \rangle \]  

(18)

The full and zoomed versions of the corresponding value sets for \( \omega = 0.005:3 \) and \( \Theta = 9:0.01:11 \) are in Figs. 9 and 10, respectively. The zero point is excluded from the value sets, the family has a stable member and so it is robustly stable.

V. CONCLUSION

The paper has been focused on application of very universal graphical approach, based on plotting the value sets of quasi-polynomials and use of the zero exclusion condition, to robust stability analysis for feedback control loops with fractional order time-delay plants and integer order PID controllers. The illustrative examples present the typical shapes of the value sets for the cases of uncertain gain, uncertain time constant and uncertain time-delay term. The future research should deal with the issue in more detail and verify e.g. the cases with more uncertain parameters together.

REFERENCES


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