Modeling the Value at Risk (VaR) of Energy Commodities Futures Using Extreme Value Copulas

Xue Gong and Songsak Sriboonchitta

***Abstract*—** This study use the Extreme Value Copula to construct the joint distribution, which is adopted to estimate Value at Risk (VaR) of a portfolio consisting of the crude oil and natural gas commodities futures. When the VaR estimation focus on modelling extreme values, i.e., the tails of the distribution, the extreme value copula may be a good choice, since it considers max-stable distributions and give certain restrictions on the copulas. The heavy tail distribution has been found in crude oil margin and the thin tail is detected in natural gas margin. Moreover, we estimate VaR of the underlying portfolio at 90% and 95% by out of sample forecasting. According to the results of backtesting, we compare the out-of-sample forecasting performance of VaR by several extreme value copulas and benchmark method. The results show that the extreme value copulas have out-of-sample forecasts than the benchmark one.

Keywords: Extreme value copula, Value at Risk, Energy commodity futures, Risk management.

I. Introduction

THE copula method has been used widely in modelling the dependence of the financial assets. In the application, the most interested part is to model the largest expected loss, i.e. the extreme value in the market; therefore the extreme value copula which is used as a tail dependence modelling may be a good choice.

Value at Risk (VaR) is one of most widely used measures in financial risk management. This measure gives a threshold loss such that the probability that the loss on the portfolio over the given time horizon exceeds this value is p. The advantage of VaR is that it reduces the risk to just one single number (Jorion, 2007). It is simple and also useful. There are many methods to estimate the VaR, but they are mainly categorized in three groups: (1) parametric method, (2) non-parametric method and also (3) semi-parametric method. The method we introduce here is the parametric method, which makes specific distributional assumptions on returns, i.e., the extreme value distribution and then calculates the corresponding VaRs.

Moreover, the commodity futures, such as the energy futures always exhibit heavy-tailed. As we known, the financial asset returns has two kinds of non-normal features the joint distribution and the distribution of margin, both of them exhibit the heavy tail and extreme tail dependence. The characteristic of the extreme movement can be captured by different models (Bastianin, 2009 [1]). The alternative distributions are student t distribution, which focuses on the heavy tail, or skewed student t distribution, which focuses on skewness and heavy tail (As Lu, Lai, and Liang, 2011 [2]). However, in this study we select the extreme value approach since it models directly on the tails of the distribution, and more flexible than the student t distribution or skewness t distribution. Moreover, the extreme value copulas are the copulas which can connect the component-maxima margins. It could be a promising approach to model the VaR of portfolio.

In our study, we use the extreme value copula with component maxima margins to estimate VaR of a portfolio which consist of crude oil futures and natural gas futures traded on the New York Mercantile Exchange (NYMEX). Since the relationship between the oil and natural gas is interacted, it is interesting and meaningful to investigate the dependence of them. The main objective is to investigate the VaR of the diversification portfolio consisting of two energy commodities.

The rest of the study is organized as follows. Section 2 introduces the theory of copulas and extreme value copula modelling. Section 3 illustrates how to use copulas to model VaR by out-of-sample forecasts. Empirical results are presented in Sect. 4. Section 5 concludes.

II. Extreme Value Copula

A. Copulas

Copula is a useful tool to link univariate distribution functions to a multivariate probability distribution. Copulas are used widely in financial risk management, especially in credit scoring, derivative pricing, and portfolio selection (Rootzén, and Tajvidi, 1997 [3]; Poon., Rockinger, and Tawn, 2004[4]).

A two-dimensional copula is a distribution function $[0, 1]^2$ with standard uniform marginal distributions. The copula for every $(u_1, u_2) \in [0,1]^2$ can be expressed as

$$\begin{aligned} C(u_1,u_2) &= P[F_1(X_1) \le u_1, F_2(X_2) \le u_2] \\ &= P[X_1 \le F_1^{-1}(u_1), X_2 \le F_2^{-1}(u_2)] \\ &= F[F_1^{-1}(u_1), F_2^{-1}(u_2)] \end{aligned} \quad (1)$$

Theorem (Sklar 1959 [5]). Let F be a joint distribution function with margins $F_1, \ldots, F_d$. Then there exists a copula C: $[0, 1]^2 \rightarrow [0, 1]$ such that, for x_1 and x_2,

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$$F(x_1, x_2) = C(F_1(x_1), F_2(x_2)) \quad (2)$$

If F_1 and F_2 are both continuous, then C is uniquely defined.

This theorem implies that every multivariate distribution has and only has one copula, and the combination of copulas with univariate distribution function can be used to obtain multivariate distribution functions (Gudendorf, and Segers, 2010 [6]; Cebrian, Denuit, and Lambert, 2003 [7]).

B. The Extreme Value Copulas

The commodities futures suffered the extreme co-movement, especially in the energy futures since the crude oil and natural gas are substitutes and also complements in consumption and production. When the demand or supply is tight (loose), the price will shoot high (low) together. Therefore, it is reasonable to study these two futures by extreme value Copula. They are one kind of copulas, which are the possible limits of copulas of component-wise maxima of i.i.d. samples.

Consider two series of component-wise maxima:

$$M_n = \max(X_1, ..., X_n) \quad (3)$$

$$N_n = \max(Y_1, ..., Y_n) \quad (4)$$

Assume that the pairs (X_i, Y_i) are independent and that their common bivariate distribution function is H with marginal distribution functions F_1 and F_2 as in (1). Then the distribution functions of M_n and N_n are:

$$\Pr(M_n \le x) = F^n(x) \quad (5)$$

$$\Pr(N_n \le y) = G^n(y) \quad (6)$$

The joint distribution of two series is:

$$\Pr[M_n \le x, N_n \le y] = H^n(x, y) \quad (7)$$

The extreme value copula has the maxima-stable property, which said that from the extreme value (maxima) we can derive the whole joint distribution.

C. The Pickands dependence function

A copula C is called as an extreme-value copula where there is a real-valued function A on the interval [0, 1] such that

$$C(u, v) = \exp\left\{\log(uv) A\left(\frac{\log(v)}{\log(uv)}\right)\right\} \quad (8)$$

for $0 < u, v < 1$, $A:[0,1] \to [1/2, 1]$ is convex and satisfies $k \vee (1-k) \le A(k) \le 1$ for all $k \in [0,1]$ (Gudendorf, and Segers, 2010 [6]). Specially, $A(0) = A(1) = 1$.

D. The extreme value copula families

There are several extreme value copulas, they are:

Gumbel copula

The dependence function is

$$A(w) = [(1-w)^r + w^r]^{1/r}$$

with $r \ge 1$. The corresponding copula function is given by

$$C(u_1, u_2) = \exp\left\{-[(-\ln u_1)^r + (-\ln u_2)^r]^{1/r}\right\} \quad (9)$$

when r=1, it means independence, when $r = \infty$, it approaches to complete dependence.

Husler-Reiss (HR) copula

the HR copula has following corresponding distribution:

$$C(u_1, u_2) = \exp\left\{\Phi\left[\frac{a}{2} + \frac{1}{a}\ln\left(\frac{\ln u_2}{\ln u_1}\right)\right]\ln u_1 + \Phi\left[\frac{a}{2} + \frac{1}{a}\ln\left(\frac{\ln u_1}{\ln u_2}\right)\right]\ln u_2\right\} \quad (10)$$

where Φ is the standard normal cumulative distribution function.

Galambos copula (negative logistic model)

The dependence function:

$$A(t) = 1 - \{t^{-1/\theta} + (1-t)^{-1/\theta}\}^{-\theta}$$

and the corresponding distribution is

$$C(u_1, u_2) = u_1 u_2 \exp\left\{-\left((-\log u_1)^{-\theta} + (-\log u_2)^{-\theta}\right)^{-1/\theta}\right\} \quad (11)$$

where $\theta > 0$.

E. The estimation problem

There are two steps to estimate the extreme value copulas (Larsson, 2010 [8]):

Step one: to estimate the marginal distribution function F_n and G_n of M_n and N_n.

Step two: to estimate the copula C_n.

F. The goodness of fit test for the copula

To choose an appropriate copula is critical (Durrleman, et al., 2000 [9]; Liu and Sriboonchitta, 2013 [10]). One of methods is to find the copula which is to minimize the distance between the empirical copula and the proposed copula. Another criterion is to measure AIC and BIC. The last one we introduced here is the goodness of fit (GOF) tests (Genest et al., 2009 [11]). Although there are many kinds of GOF tests, we will use the Cramérvon Mises (CVM) statistic which is simple and also powerful.

$$S_n = \sum_{t=1}^{n}\left\{C_k(u_t, v_t; \hat{k}) - C_n(u_t, v_t)\right\}^2 \quad (12)$$

This measures the distance between the fitted copula $C_k(u_t, v_t; \hat{k})$ and the empirical copula Cn.

G. The multivariate VaR of the portfolio

The VaR of the univariate asset is actually a quantile. The definition is as follows (Embrechts and Puccetti, 2006 [12]):

For $\alpha \in [0,1]$, at probability level α for a random variable Y, that is.

$$VaR_\alpha(Y) = \inf\{x \in \mathbb{R} : G(x) \ge \alpha\} \quad (13)$$

It should be noted that when G is strictly increasing function, $VaR_\alpha(Y)$ is the unique threshold t at which $G(t) = \alpha$. However, with the multivariate marginal, there are infinite vectors $s \in \mathbb{R}^k$ at which $G(s) = \alpha$. Therefore, the multivariate VaR at probability level α for an increasing function G is a set:

$$VaR_\alpha(G) = \partial\{x \in \mathbb{R}^k : G(x) \ge \alpha\} \quad (14)$$

According to Denuit (1999) [13], the VaR associate with S = X_1+ X_2 will lie within the bounds:

$$VaR_\alpha(G) = \partial\{(X_1, X_2) \in \mathbb{R}^2 : G(X_1 + X_2) \ge \alpha\} \quad (15)$$

$$\partial\{G(X_1 + X_2^*) \ge \alpha\} \le \partial\{G(X_1 + X_2) \ge \alpha\} \le \partial\{G(X_1 + X_2^{**}) \ge \alpha\} \quad (16)$$

The aggregate risks in which the variable X_2^* and X_2^{**} are both distributed as X_2 but are respectively in perfect negative and positive dependence with X_1 via the relation:

$X_2^* = F_2^{-1}\{1 - F_1(X_1)\}$, $X_2^{**} = F_2^{-1}\{F_1(X_1)\}$

where Fi is the distribution function of Xi and

$F_i^{-1}(t) = \inf\{s \in \mathbb{R} : F_1(s) \geq t\}, \quad i = 1, 2$

It is obvious that the VaR lies in the boundary of totally dependent and totally independent case, the value depends on the dependence degree of two assets.

III. EMPIRICAL STUDY

A. Data Description

We examine the VaR of the portfolio of two commodities futures: crude oil and natural gas futures traded on the NYMEX. The weekly closing futures prices are collected, which is covering the period of January 7, 2005 to January 2, 2015, totally 552 observations, 11 years. The data are sourced from Datastream. The percentage returns are adopted in changes in log of prices, that is, $\log(p_t/p_{t-1})$x100. The descriptive statistics of the two price returns are shown in the Table.1.

It should be noted that the returns of oil is higher than the natural gas, however, the standard deviation is lower. The correlation between these two products is 0.289. The skewness of oil is negative while natural gas is positive. The oil series exhibit much higher excess kurtosis. The Jarque-Bera statistic also confirms that that the series are not normal distribution.

Table.1 The Summary Statistics

	Crude Oil	Natural Gas
Min	-21.045	-21.604
Max	18.598	22.852
Mean	0.044	-0.135
Median	0.327	-0.232
St.dev	4.028	6.469
Skewness	-0.612	0.126
Kurtosis	3.067	0.547
JB statistics	240.04(***)	8.229(***)
No. of observations	521	521
Correlation	0.289	

B. Modelling the dependence between the futures commodities

The results of both margin and dependence are shown in Table. 2. Since we use the one-step method, we present the two margins in all of the four copula models. We can see that the estimated parameters of GEV margins are consistent with each other. The shape parameter of crude oil is positive, while the natural gas is negative. That implies that the Oil future exhibits the heavy-tailed, while the Natural gas is thin-tailed. This result justifies the use of the GEV distribution, which can measure different shapes of tails. The Tawn copula has the best fit in in-sample analysis according to AIC. The Kendall tau of the Tawn copula is around 0.1.

Table 2 The Estimation Results of Four Extreme Value Copulas

	Gumbel Copula		Galambos Copula		Husler-Reiss Copula	
	Oil	gas	Oil	gas	Oil	Gas
mu	4.697	9.719	4.699	9.717	4.699	9.717
	(0.518)***	(1.097)***	(0.517)***	(1.098)***	(0.517)***	(1.097)***
beta	2.221	4.21	2.22139	4.208	2.221	4.208
	(0.376)***	(0.846)***	(0.376)***	(0.846)***	(0.376)***	(0.846)***
xi	0.093	-0.004	0.092	-0.005	0.092	-0.005
	(0.122)	(0.246)	(0.122)***	(0.246)***	(0.122	(0.246)***
r	(1.013)**		(0.061)*		(0.212)***	
AIC	253.549		253.557		253.557	

C. The Goodness of Fit Test

Table 3 Cramér–von Mises Statistics

	Gumbel	Galambos	Husler-Reiss
statistics	0.0343	0.0254	0.0212
p-value	0.467	0.513	0.528

Note: The p-value was obtained by using a boots tapping process.

In table 3, the CVM statistic and its corresponding P-value are presented. All of the three copula models are not reject the null hypothesis, therefore the three copulas are all proper for our study.

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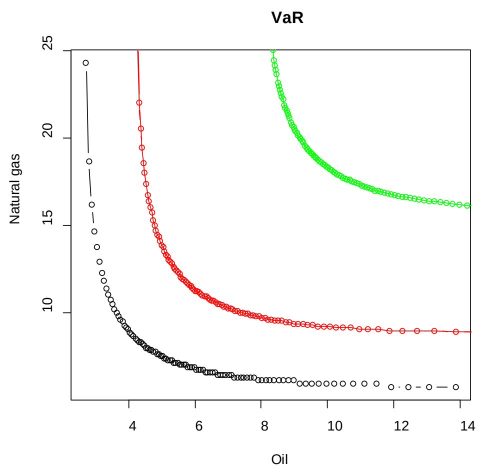
D. *The in-sample VaR analysis*



Note: the black mix line is VaR0.90, the red mixed line is VaR0.95, and the green mixed line is VaR0.99.

To estimate the multivariate VaR, we used the results of Gumbel Copula in in-sample analysis since it has the smallest AIC. The following steps are conducted, to obtain $VaR_{0.90}$ for the bivariate distribution. First, we give the first margin oil price as the 90% quantile of F_1, with the GEV distribution $0.9^{(24)}$ quantile (the observations in one block is 24), the VaR for the first margin is 2.72.

Second, to keep the quantile of bivariate distribution as 90%, by using the numerical method, we obtain the second margin 5.804, which is almost 99.9% quantile of F_2.

Third, repeat the first step and second step 100 times with accumulated 0.01 quantile of F_1 each time. That is, start from 90%, 91%, 92% quantile, until 99% quantile, we make 100 points and then draw the curves, as Fig.1.

Fig.1 shows that the lowest curve is $VaR_{0.90}$ for bivariate risk, and the higher curve is bivariate $VaR_{0.95}$. The top curve is the bivariate $VaR_{0.99}$.

In our study, the bivariate portfolio VaR is the sum of two margins such that the probability of bivariate distribution is equal to q. For the bivariate $VaR_{0.90}$, it is between [12.48, 27.01]. For the bivariate $VaR_{0.95}$, it is between [17.19, 31.15], and the last for $VaR_{0.99}$ of the bivariate distribution is between [28.21, 34.90]. Therefore, we receive a range of VaR, which has the worst and best situation. In our case, the VaRs is not much different than the independent copula, since the dependency parameter is quite small, the dependence is weak.

E. *The Out-of-Sample VaR Forecasts*

The out of sample are from the last three years of our data set, which is from January 2, 2012 to January 2, 2015, totally 144 observations. We use the 377 rolling window span to do forecasting, that is, drop first observation and add another latest observation. Therefore, we totally get 144 forecasting points. We use the violation rate to measure the performance of four extreme value copula. The benchmark method to estimate the VaR is the historical method, which is the nonparametric method. As same as the in-sample analysis, we fix the first margin to some level, such as for $\alpha = 0.90$, we fix the first margin as 0.95 quantile, and get the second margin, then sum them up. The results are shown in the Table 4. The backtesting method shows that comparing to the historical method, the Gumbel copula has better forecasting ability at 0.9 level. however, for the VaR0.95, it is not clear which method is better, this is because our data set is small. And also for our extreme value copula, the GEV margins are from each 24 observations; therefore the VaR is not that flexible and frequently change.

Table 4 The Violation Rate of Multivariate VaR

	Expected Violation	Gumbel Copula	Historical Method
VaR0.9	0.1	0.03	0.001
VaR0.95	0.05	0.00	0.00

IV. Conclusions

In this study, we present multivariate VaR of portfolio which consists of crude oil and natural gas futures by using the extreme value copula. It is a good tool to estimate the VaR due to the fact that extreme value copulas also specifically focus on the tail distribution and tail dependence.

Our out-of-sample results may be not strong evidence to prove that the extreme value copulas are superior to the other method, since the data set is small. The future work should use longer data span to verify it. The multivariate VaR measures also can be improved according to the literatures, such as the method in Embrechts, Höing, and Juri (2003) [17].

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