Bounded Control based on Norm Differential Game for Three-Player Conflict

Mao Su, Yongji Wang, and Lei Liu

***Abstract*—**A Three-player conflict with bounded controls is considered in this paper. Optimal pursuit-evasion strategy based on differential game with bounded control is derived assuming that the attacker, target and it's defender have linearized kinematics, arbitrary-order linear adversaries' dynamics, and perfect information. The obtained strategy is dependent on the zero-effort miss distance of two pursuer-evader pairs: attacker with target and defender with attacker. For adversaries with first-order dynamics, this paper presents algebraic conditions for the three-player problem and gives an analytical hybrid strategy for the attacker to capture the target while evading the defender is obtained. The simulation results shows that, by using the hybrid strategy, the attacker will evade form the defender successfully and guarantee the miss distance from the target.

***Keywords*—**Bounded control, Norm differential game, Three-player conflict, Optimal strategies.

I. INTRODUCTION

IN this paper, a differential game of three players with bounded controls is investigated. In this game, an attacker denoted by M pursues a target denoted by T, who has a defender denoted by D to protect itself from the attacker. Differential game theory is a natural setup to discuss problems such as this one[1]. The most common pursuit and evasion game called "zero-sum differential game" deals with two vehicles with respect to miss distance. In generating guidance laws, a common practice is to linearize with respect to collision course, which implies linearized kinematics. There are two formulations[2]: the first is the "linear quadratic differential game"(LQDG), and the second is the "Norm differential game"(NDG). In the LQDG, the controls are unbounded, the cost function is the weighted sum of three quadratic terms: the square of the miss distance and two penalty terms: the integrals of the respective control energy of the players[3, 4]. The optimal solution of this formulation is linear. In the NDG[5], the controls have hard bounds and the cost is purely terminal to account for imposed on the miss distance. Contrary to the LQDG, the optimal strategies are nonlinear, at a certain time before termination, the guidance law becomes pure bang-bang. The typical engagement is a one-side engagement of missile and aircraft, however in these days, a missile-missile engagement is concerned, particularly an exoatmospheric engagement. In this scenario, the guaranteed miss distance for the interceptor must be very small for hit-to-kill, especially for evasive maneuvers.

Recently, an interest in defending aircraft from an attacking missile is produced. In such a scenario, the target launching a defending missile to protect itself from an attacking missile, it is a three-body pursuit and evasion problem. In[6], a closed-form relation was derived for the initial missile–target range ratio, under the assumption of a constant collision course. In[7] the three-body game was presented as a two-team dynamic game: the lady, the bandit, and the bodyguard. The objective of the bandit is to capture the lady, while the objective of the lady and her bodyguard is to prevent this. Perelman[8] has presented a cooperative target–defender guidance strategy based on a two team LQDG against a pursuing missile provided an optimal analytic solution for the target–defender pair. Shaferman[9] and Shima[10] have presented a multiple-model adaptive guidance strategy to defend the target from the missile. Yamasaki[11], Ratnoo[12] and Shima[13] have also made some noticeable contributions on this problem. However, the obtained guidance laws in these papers are still linear and suffer the same drawbacks mentioned before. Contrary to the research before, Rubinsky and Gutman[14, 15] have focused on the strategies for the attacker and gave algebraic conditions for the attacker to capture the evader while evading the defender.

In this paper, a three-player conflict problem has been investigated based on differential game with bounded controls, an optimal strategy has been derived. Under assumption of first-order dynamics, linear kinematics and perfect information, the optimal strategies for the attacker, target and defender are obtained. Moreover, for the attacker a useful strategy based on differential game has been proposed to perform an evasion maneuver with respect to the defender, without losing its pursuit capabilities and finally an analytical solution for the game has been obtained.

This paper is organized as follows: the engagement formulation of this problem is outlined in Section II; a simple solution for the three-player game is derived and some simulation results are shown in section III; and a conclusion follows in Section IV.

II. ENGAGEMENT FORMULATION

In this problem, there are three entities: an evading target denoted as T, an attacker for intercepting the target denoted as

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M, and a defender for protecting the target denoted as D. The strategy of each player in the game is known by others. The target launches the defender to intercept the incoming attacker. It is obviously that, to protecting the target the engagement between the defender and attacker should be planned to terminate before that between the attacker and the target. On the other hand, the attacker should evade the defender and pursue the target by appropriate strategy. In this section we first present the nonlinear kinematic equations of the problem, then we will present the linearized equations used for the optimal strategies derivation. The engagement is analyzed and simulated in two dimensions.

A. Nonlinear Kinematics

A schematic view of the three-body terminal engagement geometry in the planar is shown in

Fig. 1. The notations MT and MD denote the attacker with target and attacker with defender respectively. The speed, normal acceleration and flight-path angles are denoted by V, a and γ, respectively. The range between the adversaries is r, and λ is the angle between the line of sight (LOS) and the reference frame axis.

Also given the hard bounds on the accelerations,

$$\left|a_M\right| \le a_M^{\max},\ \left|a_D\right| \le a_D^{\max},\ \left|a_T\right| \le a_T^{\max} \tag{1}$$

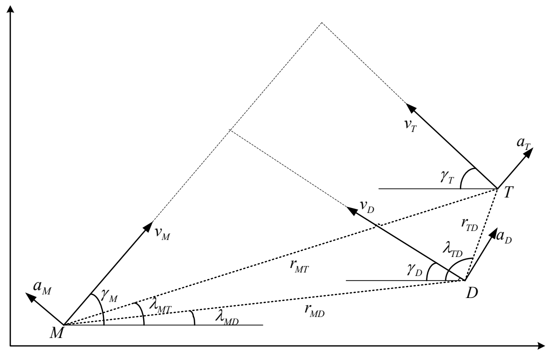

Fig. 1 Target-attacker-defender engagement geometry.

Neglecting the gravitational force during the terminal guidance phase, the engagement kinematics between the attacker and the target can be expressed as:

$$\dot{r}_{MT} = -V_M \cos\left(\gamma_M - \lambda_{MT}\right) - V_T \cos\left(\gamma_T + \lambda_{MT}\right) \tag{2}$$

$$\dot{\lambda}_{MT} = \left[V_T \sin(\gamma_T + \lambda_{MT}) - V_M \sin(\gamma_M - \lambda_{MT})\right] / r_{MT} \tag{3}$$

Similarly, the engagement kinematics equations between the defender and the attacker are

$$\dot{r}_{MD} = -V_M \cos\left(\gamma_M - \lambda_{MD}\right) - V_D \cos\left(\gamma_D + \lambda_{MD}\right) \tag{4}$$

$$\dot{\lambda}_{MD} = \left[V_D \sin(\gamma_D + \lambda_{MD}) - V_M \sin(\gamma_M - \lambda_{MD})\right] / r_{MD} \tag{5}$$

During the terminal guidance phase, the speeds of the adversaries are assumed constant.

The path angles of the adversaries are

$$\dot{\gamma}_i = a_i / V_i \quad i = \{M, T, D\} \tag{6}$$

The dynamics of each agent during the endgame can be represented by arbitrary-order linear systems:

$$\begin{cases} \dot{\boldsymbol{x}}_i = A_i \boldsymbol{x}_i + \boldsymbol{B}_i u_i \\ a_i = \boldsymbol{C}_i \boldsymbol{x}_i + d_i u_i \quad i = \{M, T, D\} \\ \left|u_i\right| \le \rho_i \end{cases} \tag{7}$$

where $\boldsymbol{x}_i$ is the state vector of an player's internal state variables with dim($\boldsymbol{x}_i$)=n_i and u_i is the controller.

B. Linearized Kinematics

During the terminal guidance phase, we have two collision triangles: one is the scenario between the attacker and target and the other in the scenario between the defender and the attacker. We can assume that the trajectories of the three bodies can be linearized near the collision triangles.

Denote y_{MT} as the relative displacement between the attacker M and the target T, normal to the initial LOS is denoted as LOS$_{MT0}$ of attacker and the target; similarly, y_{MD} is the relative displacement between M and D, normal to LOS$_{MD0}$, which is the initial LOS attached with M and D.

It is worth noting that in the engagement problem, a, the acceleration, is perpendicular to the LOS. The controller u_M, u_T and u_D are also normal to the corresponding LOS.

The state vector of the linearized problem is

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_{MT}^T & \boldsymbol{x}_M^T & \boldsymbol{x}_{MD}^T \end{bmatrix}^T \tag{8}$$

where

$$\boldsymbol{x}_{MT} = \begin{bmatrix} y_{MT} & \dot{y}_{MT} & \boldsymbol{x}_T^T \end{bmatrix}^T \tag{9}$$

$$\boldsymbol{x}_{MD} = \begin{bmatrix} y_{MD} & \dot{y}_{MD} & \boldsymbol{x}_D^T \end{bmatrix}^T \tag{10}$$

and dim($\boldsymbol{x}$)=4+n_M+n_T+n_D.

The state x_1 and x_{n_T+3} are the differences between the target and the attacker positions and between the attacker and the defender positions normal to the initial line of sight; x_2 and x_{n_T+4} are therefore the relative respective lateral speeds, and their derivatives are the relative lateral accelerations of attacker-target and attacker-defender. Thus, the equations of motion that represent the engagement's kinematics can be written as

$$\dot{\boldsymbol{x}} = \begin{cases} \dot{\boldsymbol{x}}_{MT} = \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = a_T - a_M \\ \dot{\boldsymbol{x}}_T = \boldsymbol{A}_T \boldsymbol{x}_T + \boldsymbol{B}_T u_T \end{cases} \\ \dot{\boldsymbol{x}}_M = \boldsymbol{A}_M \boldsymbol{x}_M + \boldsymbol{B}_M u_M \\ \dot{\boldsymbol{x}}_{MD} = \begin{cases} \dot{x}_{n_T+n_M+3} = x_{n_T+n_M+4} \\ \dot{x}_{n_T+n_M+4} = a_M - a_D \\ \dot{\boldsymbol{x}}_D = \boldsymbol{A}_D \boldsymbol{x}_D + \boldsymbol{B}_D u_D \end{cases} \end{cases} \tag{11}$$

The vector form of these equations can be written as

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\begin{bmatrix} u_T & u_D \end{bmatrix}^T + \boldsymbol{C} u_M \tag{12}$$

C. Timeline

The initial range between the attacker and the target is r_{MT0}. Similarly, the initial range between the attacker and the defender is r_{MD0}. Under the linearization assumption of small deviations from a collision triangle, the interception time is fixed, satisfying

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$$t_{fMT} = -r_{MT_0} / \dot{r}_{MT_0} = -r_{MT_0} \Big/ \Big[V_M \cos\left(\gamma_{M_0} + \lambda_{MT_0}\right) + V_T \cos\left(\gamma_{T_0} - \lambda_{MT_0}\right) \Big] \tag{13}$$

Similarly,

$$t_{fMD} = -r_{MD_0} / \dot{r}_{MD_0} = -r_{MD_0} \Big/ \Big[V_M \cos\left(\gamma_{M_0} + \lambda_{MD_0}\right) + V_D \cos\left(\gamma_{D_0} - \lambda_{MD_0}\right) \Big] \tag{14}$$

We define Δt as the time difference between interceptions,

$$\Delta t = t_{fMT} - t_{fMD} \tag{15}$$

and we require that the *MD* engagement terminates before that of *MT*. Thus, Δt>0, and the defender ceases to exist after t=t_{fMD}, we enforce $u_D = 0 \forall t \geq t_{fMD}$.

D. Cost Function

The interception scenario can be considered as a zero-sum differential game for the system(12) with bounded controls(1). We define two miss distance as the norm cost functions

$$J_{MT} = \left\| Z_{MT}\left(t_{fMT}\right) \right\| \tag{16}$$

$$J_{MD} = \left\| Z_{MD}\left(t_{fMD}\right) \right\| \tag{17}$$

The problem involving the three agents is posed as a norm differential game between two teams. One team is composed of the target and its defender to maximize the cost function J_{MD} and minimize J_{MT} , and the attacker belongs to the other team to minimize J_{MT} and maximize J_{MD} . Thus ,we can rewrite the cost functions as

$$\begin{aligned} &\max_{[u_T \quad u_D]} \min_{u_M} J_{MT} \\ &\min_{[u_T \quad u_D]} \max_{u_M} J_{MD} \end{aligned} \tag{18}$$

E. Order Reduction

To simplify the solution and to reduce the problem's order, we will use the following transformation by introducing a new variable:

$$\begin{cases} Z_{MT}(t) = \boldsymbol{D}_{MT} \boldsymbol{\Phi}\left(t_{fMT}, t\right) \boldsymbol{x}(t) \\ Z_{MD}(t) = \boldsymbol{D}_{MD} \boldsymbol{\Phi}\left(t_{fMD}, t\right) \boldsymbol{x}(t) \end{cases} \tag{19}$$

where $\boldsymbol{\Phi}$ is the transition matrix associated with Eq.(12), and $\boldsymbol{D}_{MT}$, $\boldsymbol{D}_{MD}$ are constant vectors,

$$\begin{cases} \boldsymbol{D}_{MT} = \left[1 \quad [0]_{1\times(n_T+n_D+n_M+3)}\right] \\ \boldsymbol{D}_{MD} = \left[[0]_{1\times(n_T+n_M+2)} \quad 1 \quad [0]_{1\times(n_D+1)}\right] \end{cases} \tag{20}$$

and $Z_{MT}(t)$ and $Z_{MD}(t)$ define the zero-effort miss(ZEM) vector:

$$\boldsymbol{Z}(t) = \left[Z_{MT}(t) \quad Z_{MD}(t)\right]^T \tag{21}$$

The derivative with respect to time of Z_{MT} is

$$\begin{aligned} \dot{Z}_{MT}(t) &= \boldsymbol{D}_{MT}\left[\dot{\boldsymbol{\Phi}}\left(t_{fMT}, t\right)\boldsymbol{x} + \boldsymbol{\Phi}\left(t_{fMT}, t\right)\dot{\boldsymbol{x}}\right] \\ &= \boldsymbol{D}_{MT}\boldsymbol{\Phi}\left(t_{fMT}, t\right)\left(\boldsymbol{B}\left[u_T \quad u_D\right]^T + \boldsymbol{C}u_M\right) \end{aligned} \tag{22}$$

Similarly,

$$\begin{aligned} \dot{Z}_{MD}(t) &= \boldsymbol{D}_{MD}\left[\dot{\boldsymbol{\Phi}}\left(t_{fMD}, t\right)\boldsymbol{x} + \boldsymbol{\Phi}\left(t_{fMD}, t\right)\dot{\boldsymbol{x}}\right] \\ &= \boldsymbol{D}_{MD}\boldsymbol{\Phi}\left(t_{fMD}, t\right)\left(\boldsymbol{B}\left[u_T \quad u_D\right]^T + \boldsymbol{C}u_M\right) \end{aligned} \tag{23}$$

Note that in Eq.(22) and Eq.(23), it is reasonable to find the explicit form of the ZEM variables. Consider the transition matrix from Eq.(19):

$$\dot{\boldsymbol{\Phi}}\left(t_f, t\right) = -\boldsymbol{\Phi}\left(t_f, t\right)\boldsymbol{A},\ \boldsymbol{\Phi}\left(t_f, t_f\right) = \boldsymbol{I} \tag{24}$$

Change the running time t to the time to go t_{go}

$$\begin{aligned} &t \to t_f - t = t_{go},\ \ dt \to -dt_{go} \\ &\dot{\boldsymbol{\Phi}}\left(t_{go}\right) = \boldsymbol{\Phi}\left(t_{go}\right)\boldsymbol{A},\ \ \boldsymbol{\Phi}(0) = \boldsymbol{I} \end{aligned} \tag{25}$$

Thus, we can obtain an equivalent reduced-order problem of finding the optimal pursue and evasion strategy for the target, it's defender and the attacker respectively. The cost function is Eq.(18) and the problem is subject to the scalar dynamics of Eq.(22)-Eq.(23) and the constraints of Eq.(1).

F. Simple Solution for Three-player Game

The defined engagement problem could be divided into two phases: the first phase is before the termination of the engagement between the defender and the attacker ($t < t_{fMD}$), the second phase is from that time onward ($t_{fMD} < t < t_{fMT}$).

G. General solution of the differential game

Differentiate the ZEM norm $\left|Z_{MT}(t)\right|$ and $\left|Z_{MD}(t)\right|$ and obtain

$$\frac{d}{dt}\left|Z_{MT}(t)\right| = \operatorname{sign}\left(Z_{MT}\right)\left(P_{MT}u_T + Q_{MT}u_D + R_{MT}u_M\right) \tag{26}$$

$$\frac{d}{dt}\left|Z_{MD}(t)\right| = \operatorname{sign}\left(Z_{MD}\right)\left(P_{MD}u_T + Q_{MD}u_D + R_{MD}u_M\right) \tag{27}$$

where

$$\begin{aligned} P_{MT} &= \phi_{12}d_T + \boldsymbol{\phi}_{1T}\boldsymbol{B}_T \\ Q_{MT} &= -\phi_{16}d_D + \boldsymbol{\phi}_{1D}\boldsymbol{B}_D \\ R_{MT} &= -\phi_{12}d_M + \phi_{16}d_D + \boldsymbol{\phi}_{1M}\boldsymbol{B}_M \\ P_{MD} &= \phi_{52}d_T + \boldsymbol{\phi}_{5T}\boldsymbol{B}_T \\ Q_{MD} &= -\phi_{56}d_D + \boldsymbol{\phi}_{5D}\boldsymbol{B}_D \\ R_{MD} &= -\phi_{52}d_M + \phi_{56}d_M + \boldsymbol{\phi}_{5M}\boldsymbol{B}_M \end{aligned} \tag{28}$$

The target has interest to maximize the variable

$$\frac{d}{dt}\left|Z_{MT}(t)\right|$$

with its controller $u_T(t)$. Therefore, the optimal strategy for $u_T(t)$ would be

$$u_T^*(t) = \rho_T \operatorname{sign}\left(Z_{MT}\right)\operatorname{sign}\left(P_{MT}\right) \tag{29}$$

The defender, on the other hand, will try to minimize the variable

$$\frac{d}{dt}\left|Z_{MD}(t)\right|$$

with its controller $u_D(t)$. Thus. The optimal strategy for it would be

$$u_D^*(t) = -\rho_D \operatorname{sign}\left(Z_{MD}\right)\operatorname{sign}\left(Q_{MD}\right) \tag{30}$$

There are two extreme situations to be assumed to derive the bounds for the attacker.

1) The attacker will evade the defender and ignore the target. In such a case, the strategy of the attacker would be

$$u_{Me}^* = \rho_M \operatorname{sign}(R_{MD})\operatorname{sign}(Z_{MD}) \tag{31}$$

2) The attacker will pursue the target and ignore

$$u_{Mp}^* = -\rho_M \operatorname{sign}(R_{MT})\operatorname{sign}(Z_{MT}) \tag{32}$$

III. Simple Solution for Three-player Game

The defined engagement problem could be divided into two phases: the first phase is before the termination of the engagement between the defender and the attacker ($t < t_{fMD}$), the second phase is from that time onward ($t_{fMD} < t < t_{fMT}$).

A. Example of first-order dynamics

When all three entities have first-order acceleration dynamics, the switch functions of the attacker, target and defender can be written as

$$\begin{aligned} P_{MD}\left(t_{go}^{MD}\right) &= 0 \\ Q_{MD}\left(t_{go}^{MD}\right) &= -\tau_D \psi\left(t_{go}^{MD}/\tau_D\right) \\ R_{MD}\left(t_{go}^{MD}\right) &= \tau_M \psi\left(t_{go}^{MD}/\tau_M\right) \end{aligned} \tag{33}$$

$$\begin{aligned} P_{MT}\left(t_{go}^{MT}\right) &= \tau_T \psi\left(t_{go}^{MT}/\tau_T\right) \\ Q_{MT}\left(t_{go}^{MT}\right) &= 0 \\ R_{MT}\left(t_{go}^{MT}\right) &= -\tau_M \psi\left(t_{go}^{MT}/\tau_M\right) \end{aligned} \tag{34}$$

where $\psi(\zeta) = \exp(-\zeta) + \zeta - 1$, $\tau_i \ \{i = T, D, M\}$ is the time constant of each player in the game.

Using the optimal strategies, the optimal ZEM dynamics can be computed in reverse time (time-to-go), satisfying

$$\begin{aligned} \frac{dZ^*}{dt_{go}} &= \frac{dZ^*}{dt}\frac{dt}{dt_{go}} = \Gamma \operatorname{sign}\left\{Z\left(t_{go}=0\right)\right\} \\ Z\left(t_{go}=0\right) &\neq 0 \end{aligned} \tag{35}$$

B. Game space decomposition

Consider the normalization miss distance (NZEM) z_{pe}^* in two-player game problem, the pursuer will use the optimal pursuit law and the evader will use the optimal evasion law,

For the pursuer and the evader, we assume that, the maneuverability ratio and dynamics ration, denoted as μ and ε, respectively:

$$\begin{aligned} \mu &= \rho_p / \rho_e \\ \varepsilon &= \tau_p / \tau_e \end{aligned} \tag{36}$$

Define z as the normalization zero miss distance and ξ as the normalization time-to-go.

$$z = Z / \left(\tau_e^2 \rho_e\right) \tag{37}$$

$$\xi_{go} = t_{go} / \tau_e \tag{38}$$

Therefore,

$$\frac{d\left|z_{pe}^*\right|}{d\xi_{go}} = \mu\varepsilon\psi\left(\xi_{go}/\varepsilon\right) - \psi\left(\xi_{go}\right) \tag{39}$$

For the maneuverability ratio and dynamics ration, there are several cases to be discussed.

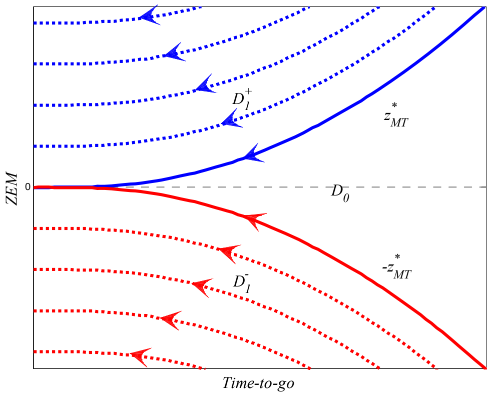

Fig. 2 Game space decomposition for pursuer-evader (case 1)

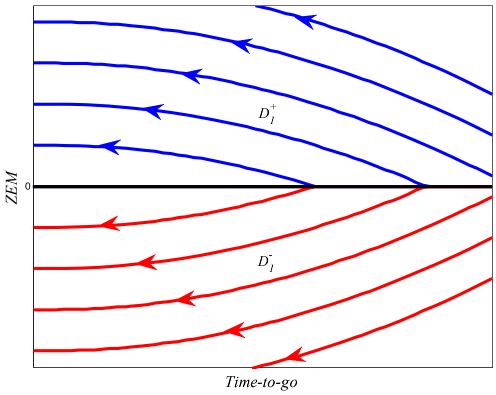

Fig. 3 Game space decomposition for pursuer-evader (case 2)

1) $\mu > 1, \varepsilon < 1$

In this case, it means that the acceleration of the pursuer is bigger than the evader and the time constant is smaller. It is obviously that, the game space decomposition is shown in Fig.2. z_{pe}^* and $-z_{pe}^*$ are the normal border trajectories and the region between the border trajectories is the singular region, in this region, any strategy used by the pursuer and the evader is optimal, as eventually these border trajectories will be reached and maintained. Thus, all initial conditions in this region will lead to a zero miss distance in the attacker-target engagement. Outside the singular region the pursuer and the evader must apply an optimal strategy.

In the region defined by $\left|z_{pe}\right| > z_{pe}^*$ we obtain, if the attacker and the target play optimal, the NZEM $\left|z_{pe}\right|$ will go parallel to $\left|z_{pe}^*\right|$.In the region between the two border trajectory z_{pe}^* and $-z_{pe}^*$, the evader and the pursuer can use arbitrary strategies to the border trajectories z_{pe}^* or $-z_{pe}^*$. We denote this region as being singular.

2) $\mu < 1, \varepsilon < 1$

In this case, it means that the acceleration of the pursuer is smaller than the evader but the time constant is smaller. The game space decomposition for pursuer-evader is shown in Fig.3. We can easily find that in this case, the miss distance increase by the time-to-go.

3) $\mu < 1, \varepsilon > 1$

In this case, the time constant of the defender is bigger than the attacker, thus, like the case 2, the miss distance will increase by the time-to-go, and the game space decomposition is the same as shown in Fig.3.

C. Normalization zero miss distance

Consider the two extreme situations mentioned in the Section II, the attacker has two optimal strategies in different situation during the first phase.

1) The attacker will use the optimal evasion strategy, therefore,

$$\frac{d\left|Z^*_{MD}\right|}{dt^{MD}_{go}} = \Gamma^*_{MD} = \rho_D \left|Q_{MD}\right| - \rho_M \left|R_{MD}\right| = \rho_D \tau_D \psi\left(t^{MD}_{go}/\tau_D\right) - \rho_M \tau_M \psi\left(t^{MD}_{go}/\tau_M\right) \tag{40}$$

2) The attacker will use the optimal pursuit strategy, therefore,

$$\frac{d\left|Z^{**}_{MD}\right|}{dt^{MD}_{go}} = \Gamma^{**}_{MD} = \rho_D \left|Q_{MD}\right| + \rho_M \left|R_{MT}\right| \operatorname{sign}\left(Z_{MT}\right)\operatorname{sign}\left(Z_{MD}\right) = \rho_D \tau_D \psi\left(t^{MD}_{go}/\tau_D\right) + \rho_M \tau_M \psi\left(t^{MD}_{go}/\tau_M\right)\operatorname{sign}\left(Z_{MT}\right)\operatorname{sign}\left(Z_{MD}\right) \tag{41}$$

Similarly, in the second phase, we have

$$\frac{d\left|Z^*_{MT}\right|}{dt^{MT}_{go}} = \Gamma^*_{MT} = -\rho_T \left|Q_{MT}\right| + \rho_M \left|R_{MT}\right| = -\rho_T \tau_T \psi\left(t^{MT}_{go}/\tau_T\right) + \rho_M \tau_M \psi\left(t^{MT}_{go}/\tau_M\right) \tag{42}$$

Integrating Eq.(35) backward from any given end condition $Z\left(t_{go}=0\right)$ generates a candidate optimal trajectory. Define the optimal border trajectory:

$$Z^*\left(t_{go}\right) = \int_0^{t_{go}} \Gamma^* d\zeta \tag{43}$$

From Eq.(43), we can easily find that the family of the optimal trajectory determines the structure of the game solution.

Under the assumption that all these three players in the endgame have similar maneuverability, we can obtain a conclusion without proving that, Γ^*_{MD} , Γ^{**}_{MD} and Γ^*_{MT} are monotonic, thus, there are three cases discussed in this paper.

For the attacker and the target, denote the maneuverability ratio μ_T and dynamics ration ε_T , respectively:

$$\begin{aligned} \mu_T &= \rho_T/\rho_M \\ \varepsilon_T &= \tau_T/\tau_M \end{aligned} \tag{44}$$

and for the defender

$$\begin{aligned} \mu_D &= \rho_D/\rho_M \\ \varepsilon_D &= \tau_D/\tau_M \end{aligned} \tag{45}$$

As mentioned in game space decomposition for two-player game, different maneuver ratio and dynamics ratio lead to different game space decomposition. In this three-player game, we assume that

$$\begin{aligned} \mu_T &< 1 \\ \mu_D &< 1 \end{aligned} \tag{46}$$

otherwise the game will terminate in the first phase and the problem will be a two-player game.

Substitute and obtain

$$\frac{d\left|z^*_{MD}\right|}{d\xi^{MD}_{go}} = \mu_D \varepsilon_D \psi\left(\xi^{MD}_{go}/\varepsilon_D\right) - \psi\left(\xi^{MD}_{go}\right) \tag{47}$$

$$\frac{d\left|z^{**}_{MD}\right|}{d\xi^{MD}_{go}} = \mu_D \varepsilon_D \psi\left(\xi^{MD}_{go}/\varepsilon_D\right) + \psi\left(\xi^{MD}_{go}\right)\operatorname{sign}\left(z_{MT}\right)\operatorname{sign}\left(z_{MD}\right) \tag{48}$$

$$\frac{d\left|z^*_{MT}\right|}{d\xi^{MT}_{go}} = -\mu_T \varepsilon_T \psi\left(\xi^{MT}_{go}/\varepsilon_T\right) + \psi\left(\xi^{MT}_{go}\right) \tag{49}$$

From Eqs.(48) it can be found that during the first phase, $\left|z^{**}_{MD}\right|$ will depend on the sign of the z_{MT} and z_{MD} ,there are two separate situations are to be discussed.

1) In the case of opposite rotation, the miss distance z_{MT} and z_{MD} have the opposite signs:

$$\operatorname{sign}\left(z_{MT}\right) = -\operatorname{sign}\left(z_{MD}\right) \tag{50}$$

From this, it is readily seen that

$$u^*_{Me} = u^*_{Mp} \tag{51}$$

Eqs.(48) can be rewrite as

$$\frac{d\left|z^{**}_{MD}\right|}{d\xi^{MD}_{go}} = \mu_D \varepsilon_D \psi\left(\xi^{MD}_{go}/\varepsilon_D\right) - \psi\left(\xi^{MD}_{go}\right) \tag{52}$$

Thus,

$$\left|z^{**}_{MD}\right| = \left|z^*_{MD}\right| \tag{53}$$

In this case, the optimal evasion law is the same as the pursuit law. It is readily seen that in both phase of the endgame, the attacker use the only one optimal strategy to evade from the defender while pursue the target. It is the simplest case because the obtained law holds for every initial condition. If the attacker has an ideal condition, when using the optimal pursuit strategy it can not only pursue the target, but also evade from the defender successfully. Case 1 is a product of initial conditions and the others' strategy, so that the attacker can't enforce it.

2) In the case of the same rotation, both line of sight rotate in the same direction, the miss distance z_{MT} and z_{MD} have the opposite signs:

$$\operatorname{sign}\left(z_{MT}\right) = \operatorname{sign}\left(z_{MD}\right) \tag{54}$$

Therefore,

$$u^*_{Me} = -u^*_{Mp} \tag{55}$$

Eqs.(48) can be rewrite as

$$\frac{d\left|z^{**}_{MD}\right|}{d\xi^{MD}_{go}} = \mu_D \varepsilon_D \psi\left(\xi^{MD}_{go}/\varepsilon_D\right) + \psi\left(\xi^{MD}_{go}\right) \tag{56}$$

Thus, there are three trajectory bounds denoted as $\left|z_{MD}^{*}\right|$, $\left|z_{MD}^{**}\right|$ and $\left|z_{MT}^{*}\right|$ in this three-player game.

D. Simulation for the three-player game

Given the player maneuver capabilities(ρ_M, ρ_T, ρ_D), the normalization time to go(ξ_{MD}, ξ_{MT}), the desired miss distance $\left|z_{MT}^{*}\left(\xi_{go}^{MT}=0\right)\right|$ and $\left|z_{MD}^{*}\left(\xi_{go}^{MD}=0\right)\right|$, and integrate Eqs.(47), Eqs.(49) and Eqs.(56) respectively, yields, the trajectory bounds are described in Fig. 4.

$\left|z_{MD}^{*}\right|$ is the evasion bound and $\left|z_{MT}^{*}\right|$ is the pursuit bound, it is essential for the attacker to keep both the of them inside the bounds, so that

$$\left|z_{MT}\left(\xi_{go}^{MT}\right)\right|<\left|z_{MT}^{*}\left(\xi_{go}^{MT}\right)\right|,\left|z_{MD}\left(\xi_{go}^{MD}\right)\right|>\left|z_{MD}^{*}\left(\xi_{go}^{MD}\right)\right| \quad (57)$$

$\left|z_{MD}^{**}\right|$ is the failsafe bound for the attacker in the case 2.

During the first phase, when the attacker use the pursuit strategy, it must guarantee the miss distance $\left|z_{MD}\right|>\left|z_{MD}^{*}\right|$, or at $\xi_{go}^{MD}=0$, the miss distance will smaller than the desired miss distance, which the attacker cannot endure.

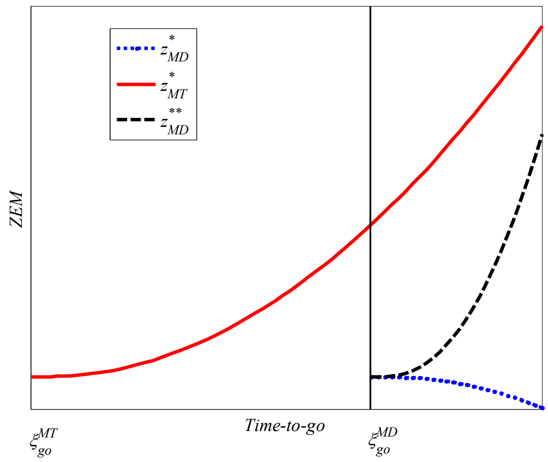

Fig. 4 Defender-attacker and attacker-target ZEM bounds

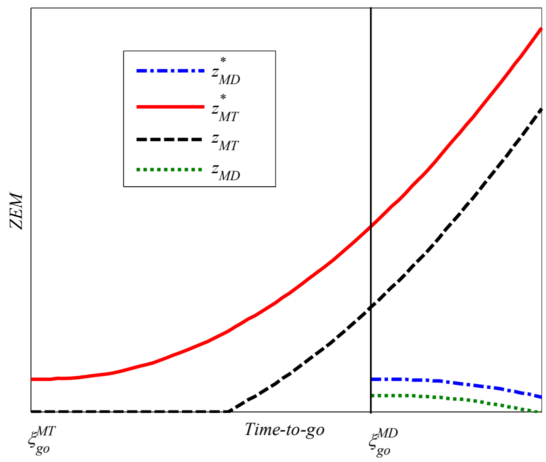

Fig. 5 Case 1 linear simulation

In the case 1 in part C in this section, when the attacker use the optimal strategy, the trajectory are both in the bounds as described in Fig. 5. The attacker can guarantee the desired miss distance and the first case is trivial.

By using the failsafe bound, the attacker can use the hybrid strategy in the first phase in the case 2. For a given desired miss distance $\left|z_{MD}^{*}\left(\xi_{go}^{MD}=0\right)\right|$, there is a failsafe bound for the attacker, so that if $\left|z_{MD}\left(\xi_{go}^{MD}\right)\right|=\left|z_{MD}^{*}\left(\xi_{go}^{MD}\right)\right|$ for any $\xi_{go}^{MD}>0$, the attacker's strategy can be safely switched form u_{Me}^{*} to u_{Mp}^{*} at that point and the miss distance of $\left|z_{MD}^{*}\left(\xi_{go}^{MD}=0\right)\right|$ can be guaranteed.

To reach $\left|z_{MD}^{**}\right|$ the attacker can use a variety of controls. Define the switch time ξ^{*}. The endgame can be divided into another two phase named as phase A, and phase B. In phase A, the attacker will use the evasion strategy (perhaps not optimal) to reach $\left|z_{MD}^{**}\right|$, while in phase B, it will use the optimal pursuit strategy to pursue the target and guarantee the miss distance.

Given the initial conditions $\left|z_0^{MD}\right|=\left|z_{MD}\left(\xi=0\right)\right|$ and $\left|z_0^{MT}\right|=\left|z_{MT}\left(\xi=0\right)\right|$, where, $\xi=t/\tau_M$. Rename some of our variables to work with a single time-to-go variable. Define

$$\xi_{go}=\xi_{go}^{MD}, \xi_f=\xi_f^{MD}, \xi_{go}+\Delta\xi=\xi_{go}^{MT}, \xi_f+\Delta\xi=\xi_f^{MT} \quad (58)$$

To verify the optimal strategy in the phase A, consider the attacker uses an evasive maneuver $u_M=a\,\mathrm{sign}\left(z_{MD}\right)$, $a\le\rho_M$ to evade the defender, as the target uses its optimal evasion law $u_T=\rho_T\mathrm{sign}\left(z_{MT}\right)$ and the defender uses its optimal pursuit law $u_D=-\rho_D\mathrm{sign}\left(z_{MD}\right)$

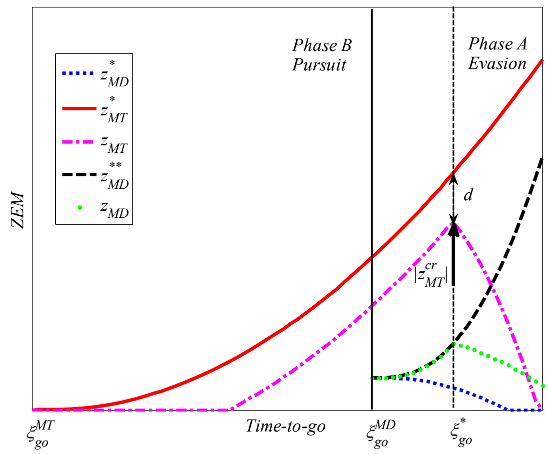

Fig. 6 Case 2 linear simulation

The $\left|z_{MT}\left(\xi_{go}\right)\right|$ equation becomes

$$\left|z_{MT}\left(\xi_{go}\right)\right|=\left|z_0^{MT}\right|+\frac{a}{\rho_M}\int_0^{\xi}\psi\left(\xi_f^{MT}-\eta\right)d\eta + \mu_T\varepsilon_T\int_0^{\xi}\psi\left(\left(\xi_f^{MT}-\eta\right)/\varepsilon_T\right)d\eta \quad (59)$$

Similarly, for the second ZEM variable,

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$$\left|z_{MD}\left(\xi_{go}\right)\right|=\left|z_0^{MD}\right|+\frac{a}{\rho_M}\int_0^{\xi}\psi\left(\xi_f^{MD}-\eta\right)d\eta \\ -\mu_D\varepsilon_D\int_0^{\xi}\psi\left(\left(\xi_f^{MD}-\eta\right)/\varepsilon_D\right)d\eta \tag{60}$$

The most aggressive strategy uses u_{Me}^* to reach the $\left|z_{MD}^{**}\right|$ and, then, switch to u_{Mp}^* .The trajectory is shown in 错误！未找到引用源。.

Define

$$\left|z_{MT}^{cr}\right|=\left|z_{MT}\left(\xi_{go}^*\right)\right| \tag{61}$$

$$d\left(\xi_{go}^*\right)=\left|z_{MT}^*\left(\xi_{go}^*\right)\right|-\left|z_{MT}^{cr}\right| \tag{62}$$

From 错误！未找到引用源。, it is readily found that, when the attacker uses pursuit law in the phase B, the miss distance $\left|z_{MT}^*\left(\xi_{go}^{MT}=0\right)\right|$ depends on the value of $d\left(\xi_{go}^*\right)$, so to find the optimal strategy, it is necessary to find the pair $\left(\xi_{go}^*,a^*\right)$ for which the value of the defined $d\left(\xi_{go}^*\right)$ is maximal in the appropriate interval. The cost is

$$d^*=\max_{\xi_{go}^*}d\left(\xi_{go}^*\right) \tag{63}$$

Define

$$l=\left|z_{MD}^*\left(\xi_{go}^{MD}=0\right)\right|,m=\left|z_{MT}^*\left(\xi_{go}^{MT}=0\right)\right| \tag{64}$$

Integration yields

$$\begin{aligned}
&\left|z_{MD}^{**}\left(\xi_{go}\right)\right|=l+\mu_D\varepsilon_D\int_0^{\xi_{go}}\psi\left(\eta/\varepsilon_D\right)d\eta+\int_0^{\xi_{go}}\psi\left(\eta\right)d\eta\\
&\left|z_{MD}\left(\xi_{go}\right)\right|=\left|z_0^{MD}\right|-\mu_D\varepsilon_D\int_0^{\xi}\psi\left(\left(\xi_f^{MD}-\eta\right)/\varepsilon_D\right)d\eta\\
&\qquad+k\int_0^{\xi}\psi\left(\xi_f^{MD}-\eta\right)d\eta\\
&\left|z_{MT}^*\left(\xi_{go}\right)\right|=m-\mu_T\varepsilon_T\int_0^{\xi_{go}+\Delta\xi}\psi\left(\eta/\varepsilon_T\right)d\eta+\int_0^{\xi_{go}+\Delta\xi}\psi\left(\eta\right)d\eta\\
&\left|z_{MT}\left(\xi_{go}\right)\right|=\left|z_0^{MT}\right|+\mu_T\varepsilon_T\int_0^{\xi}\psi\left(\left(\xi_f^{MT}-\eta\right)/\varepsilon_T\right)d\eta\\
&\qquad+k\int_0^{\xi}\psi\left(\xi_f^{MT}-\eta\right)d\eta
\end{aligned} \tag{65}$$

where $k=a/\rho_M$.

Denote $\theta(\eta)=\int\psi(\eta)d\eta$, and $\theta(0)=-1$

To find the intersection points of the functions $\left|z_{MD}\left(\xi_{go}\right)\right|$ and $\left|z_{MD}^{**}\left(\xi_{go}\right)\right|$, equate them and obtain

$$\theta\left(\xi_{go}^*\right)=\frac{\left|z_0^{MD}\right|-l-\mu_D\varepsilon_D^2\theta\left(\xi_f/\varepsilon_D\right)+k\theta\left(\xi_f\right)+\left(\mu_D\varepsilon_D^2+1\right)\theta(0)}{k+1} \tag{66}$$

or alternatively

$$k^*\left(\xi_{go}\right)=\frac{l-\left|z_0^{MD}\right|+\mu_D\varepsilon_D^2\theta\left(\xi_f/\varepsilon_D\right)-\left(\mu_D\varepsilon_D^2+1\right)\theta(0)+\theta\left(\xi_{go}\right)}{\theta\left(\xi_f\right)-\theta\left(\xi_{go}\right)} \tag{67}$$

Consider that $\theta\left(\xi_{go}\right)\ge 0$, so that

$$k\ge\frac{l-\left|z_0^{MD}\right|+\mu_D\varepsilon_D^2\theta\left(\xi_f/\varepsilon_D\right)-\left(\mu_D\varepsilon_D^2+1\right)\theta(0)}{\theta\left(\xi_f\right)} \tag{68}$$

This means that, the ratio k is bounded

$$\frac{l-\left|z_0^{MD}\right|+\mu_D\varepsilon_D^2\theta\left(\xi_f/\varepsilon_D\right)-\left(\mu_D\varepsilon_D^2+1\right)\theta(0)}{\theta\left(\xi_f\right)}=k_{\min}\le k\le 1 \tag{69}$$

If

$$\frac{l-\left|z_0^{MD}\right|+\mu_D\varepsilon_D^2\theta\left(\xi_f/\varepsilon_D\right)-\left(\mu_D\varepsilon_D^2+1\right)\theta(0)}{\theta\left(\xi_f\right)}\ge 1 \tag{70}$$

the defender can guarantee a miss distance smaller than l . The $\theta\left(\xi_{go}^*\right)$ is also bounded:

$$0\le\theta\left(\xi_{go}^*\right)\le\frac{\left|z_0^{MD}\right|-l-\mu_D\varepsilon_D^2\theta\left(\xi_f/\varepsilon_D\right)+\theta\left(\xi_f\right)+\left(\mu_D\varepsilon_D^2+1\right)\theta(0)}{2} \tag{71}$$

Furthermore,

$$d\left(\xi_{go}^*\right)=m-\left|z_0^{MT}\right|-\mu_T\varepsilon_T^2\left(\theta\left(\left(\xi_f+\Delta\xi\right)/\varepsilon_T\right)-\theta(0)\right) \\ +\left(\theta\left(\xi_f+\Delta\xi\right)-\theta(0)\right)+(k+1)\left(\theta\left(\xi_f+\Delta\xi\right)-\theta\left(\xi_{go}+\Delta\xi\right)\right) \tag{72}$$

Substituting $k^*\left(\xi_{go}\right)$ into the $d\left(\xi_{go}^*\right)$ and obtain

$$\begin{aligned}
d\left(\xi_{go}^*\right)&=m-\left|z_0^{MT}\right|-\mu_T\varepsilon_T^2\left(\theta\left(\left(\xi_f+\Delta\xi\right)/\varepsilon_T\right)-\theta(0)\right)\\
&+\left(\theta\left(\xi_f+\Delta\xi\right)-\theta(0)\right)\\
&+\frac{l-\left|z_0^{MD}\right|+\mu_D\varepsilon_D^2\theta\left(\xi_f/\varepsilon_D\right)-\left(\mu_D\varepsilon_D^2+1\right)\theta(0)+\theta\left(\xi_f\right)}{\left(\theta\left(\xi_f\right)-\theta\left(\xi_{go}\right)\right)/\left(\theta\left(\xi_f+\Delta\xi\right)-\theta\left(\xi_{go}+\Delta\xi\right)\right)}
\end{aligned} \tag{73}$$

When given the desired miss distance m , initial condition $\left|z_0^{MT}\right|$, final time ξ_f^{MT} and ξ_f^{MD}, the value of the cost function depends on the ξ_{go} .

Define

$$f\left(\xi_{go}\right)=\frac{\theta\left(\xi_f+\Delta\xi\right)-\theta\left(\xi_{go}+\Delta\xi\right)}{\theta\left(\xi_f\right)-\theta\left(\xi_{go}\right)} \tag{74}$$

Thus, to maximize the function $d\left(\xi_{go}\right)$, the function $f\left(\xi_{go}\right)$ should be maximal.

Differentiate $\theta\left(\xi_{go}\right)$ with respect to ξ_{go} , simplify, and obtain for any $\xi_{go}\ge 0$

$$\frac{d\theta}{d\xi_{go}}=\psi\left(\xi_{go}\right)=e^{-\xi_{go}}+\xi_{go}-1\ge 0 \tag{75}$$

$$\frac{d^2\theta}{d\xi_{go}^2}=-e^{-\xi_{go}}+1\ge 0 \tag{76}$$

So the function $\theta\left(\xi_{go}\right)$ is monotonic increasing and it is a concave function, the function $f\left(\xi_{go}\right)>0$ and it is also a monotonic increasing function. From this ,it is readily seen that to maximize the function $d\left(\xi_{go}\right)$,the switch time should be chosen as $\xi_{go\max}$, which corresponds to $a^*=\rho_M$.

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Finally, the strategy that maximizes $d\left(\xi_{go}\right)$ is

$$\begin{aligned}&\text{if } \operatorname{sign}\left(Z_{MD}\right)=-\operatorname{sign}\left(Z_{MT}\right)\\&u_M=u_{Me}^*\left(t\right)=u_{Mp}^*\left(t\right)\\&\text{if } \operatorname{sign}\left(Z_{MD}\right)=\operatorname{sign}\left(Z_{MT}\right)\\&u_M=u_e^*\left(t\right)\ \text{ until}\left|Z_{MD}\left(t_{go}\right)\right|=\left|Z_{MD}^{**}\left(t_{go}\right)\right|,\text{then},u_M=u_p^*\left(t\right)\end{aligned} \tag{77}$$

IV. Conclusion

Optimal evasion and pursuit strategies for a three-player conflict have been derived in this paper. This problem is analyzed for a general arbitrary-order linear dynamics, under the assumption that the information is perfect and the control is bounded, a linearized model is derived. Based on the norm differential game (NDG), a general solution for an arbitrary-order linear model is obtained, by using the first-order system for example, the optimal evasion and pursuit strategies for the attacker, the target and its defender are derived, and the game space decomposition for attacker-target and defender-attacker is analyzed, then the closed-form solution for first-order system is obtained. For the attacker, the strategies are more complicated than the target and the defender, because, it must capture the target while evading from the defender, the maneuver ratio and dynamics ratio are very important in the game. As the case 1 described in the paper, the initial conditions are very important too, but the player in the game cannot enforce it. By using the NDG method, the optimal evasion strategy for the target and the optimal pursuit strategy for the defender have been obtained. To pursue the target and evade from the defender a hybrid evasion-pursuit strategy for the attacker has been investigated, and an analytical solution is obtained. The switch time t_{go}^* is a key parameter in this problem. In this paper the optimal t_{go}^* is obtained by analysis, and the optimal switch strategy is obtained too. In the future research nonlinear kinematics would be considered, and in the real scenario, the information is not perfect, it is necessary to provide a useful estimation method considering the nonlinear kinematics of the problem, and consider the influence of the noise in the endgame.

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INVITED-DIMITROVA - Short CV of Yongji Wang

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