On the Financial Applications of Multivariate Stochastic Orderings

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Abstract—The paper proposes a multivariate comparison among different financial markets, using risk/variability measures consistent with investors’ preferences. First of all, we recall a recent classification of multivariate stochastic orderings and properly define the selection problem among different financial markets. Then, we propose an empirical financial application, using multivariate stochastic orderings consistent with the non-satiable and risk averse investors’ preferences. For the empirical analysis we examine two different approaches; first, we assume that the return are normally distributed; second, we deal with the more general assumption that the returns’ distribution follow a stable sub-Gaussian law.

Keywords—Financial Market comparison, Multivariate preferences, Stochastic Dominance.

I. INTRODUCTION

This paper focuses on the investors’ preferences related to the portfolio selection problem. Thus we introduce multivariate stochastic orderings consistent with investors’ preferences and show how we can use multivariate risk measures and orders (in terms of probability functionals) to determine dominant sectors/markets in different financial contexts.

We define the dominance among financial markets generalizing the concept of univariate FORS orderings, risk and reward measures in the multivariate framework (see Ortobelli et al. in [2], [3] and [4]). FORS probability functionals and orderings generalize those found in the literature (see [1] and [9]) and are strictly related to the theory of choice under uncertainty and to the theory of probability functionals and metrics (see [6] and [10]). While the new orderings can be used to further characterize and specify the investors’ choices and preferences, the new risk measures should be used either to minimize the risk or to minimize its distance from a given benchmark.

The main contribution of this paper is to use multivariate ordering consistent with investors’ preferences to define the dominance among financial markets/sectors. Thus we propose two different approaches: the first one is based on a generalization of the mean-variance approach. The second one takes into account the possibility of heavy tailed distributions. In this last case, the conditions for the multivariate dominance are based on a comparison between: i) means; ii) dispersion indices and iii) stability indices. Therefor we propose an ex-ante empirical application of multivariate orderings, to evaluate the possible dominance among different financial stock markets (USA, China, Japan and Germany).

The paper is organized as follows. In Section 2 we introduce multivariate FORS orderings and the definition of orderings among markets. Section 3 introduces a preliminary empirical analysis.

II. MULTIVARIATE DOMINANCE

We recall that the most important property that characterizes any probability functional associated with a choice problem is the consistency with a stochastic order.

We says that a functional \( \mu : \Lambda \times \Lambda \to R \) is consistent with a preferences orderings \( > \) anytime that \( X \) dominates \( Y \) (with respect to a given order of preferences \( > \)), implies that \( \mu(X,Z) \leq \mu(Y,Z) \) for a fixed arbitrary benchmark \( Z \) (where \( X,Y,Z \in \Lambda \), that is a non-empty space of real valued random variables defined on \((\Omega,\mathcal{F},P)\)). A univariate FORS measure induced by a given order of preferences \( > \) can be any probability functional \( \mu : \Lambda \times \Lambda \to R \) which is consistent with \( > \). Hence we can similarly define multivariate FORS measures.

**Definition 1** We call FORS measure induced by a preference order \( > \), any probability functional \( \mu : \Lambda \times \Lambda \to R^s \) (where \( \Lambda \) a non-empty set of real-valued n-dimensional random vectors defined on the probability space \((\Omega,\mathcal{F},P)\)) that is consistent with a given order of preferences \( > \) (that is, if \( X \) dominates \( Y \) with respect to a given order of preferences \( > \), implies \( \mu(X,Z) \leq \mu(Y,Z) \) for a fixed arbitrary benchmark \( Z \) where the vectorial inequality is considered for each component i.e., \( \mu_i(X,Z) \leq \mu_i(Y,Z) \) for any \( i = 1, \ldots, s \).

As for the FORS measures we can easily extend the definition of multivariate FORS ordering developed in [2] and [3].

**Definition 2** Let \( \rho_X : A \to \bar{R}^s \) (with compact and convex
A ⊆ ℝ^n) be a bounded variation function, for every n-dimensional random vector X belonging to a given class Λ. Assume that ∈ Λ, ρ_X = ρ_Y, a.e. on A iff X ≤ Y. If, for any fixed ∈ Λ, ρ_X(λ) is a FORS measure induced by an ordering >, then we call FORS orderings induced by > the following new class of orderings defined ∀ X,Y ∈ Λ: \[ \{X ∈ Λ : \int_A \prod_{i=1}^n |t_i^{a_i-1} dρ_X(t_1, ..., t_n) | < ∞ \} \] for every (a_1, ..., a_n) with a_i ≥ 1 we say that X dominates Y in the sense α-FORS ordering induced by >, in symbols:

\[ X \text{ FORS } Y >, \alpha \] if and only if \( ρ_{X,α}(u) ≤ ρ_{Y,α}(u) \) ∀u ∈ A

where

\[ ρ_{X,α}(u_1, ..., u_n) = \left\{ \frac{1}{\prod_{i=1}^n \Gamma(a_i)} \int_{a_1}^{u_1} ... \int_{a_n}^{u_n} (a_i - t_i)^{a_i-1} dρ_X(t_1, ..., t_n) \right\} \]

and the integral is a vector applied for each component of the vector dρ_X = [dρ_{(1)X}, ..., dρ_{(s)X}], whose components are the differential of components of vector ρ_X = [ρ_{(1)X}, ..., ρ_{(s)X}].

This expression generalizes the one proposed in [5]. Besides, we call ρ_X FORS measure associated with the FORS ordering of random vectors belonging to Λ. We say that ρ_X generates the FORS ordering. Multivariate orderings can have several applications in economics and finance. In this paper we discuss a possible application in ordering financial markets by the point of view of investors. With this aim we need to give some possible alternative definitions of orderings among financial markets/sectors.

Let us assume there are two markets: market A with n assets, and market B with s assets. Assume that the vector of the positions taken by an investor in the n risky assets of market A is denoted by x = [x_1, ..., x_n] and similarly the vector of the positions taken by an investor in the m risky assets of market B is denoted by y = [y_1, ..., y_s]. We assume that no short sales are allowed.

**Definition 3** We say that a market/sector A with n assets strongly dominates another market/sector B with s assets with respect to a multivariate FORS ordering if for any vector x of returns Y_B \[ t ≤ u = \min(s,n) \] assets of market/sector B there exists a vector X_A of market/sector A such that X_A FORS Y_B. Similarly, we say that a market/sector A with n assets weakly dominates another market/sector B with s assets with respect to the FORS ordering if for any given portfolio of gross returns Y of market/sector B there exists a portfolio X_A of the market/sector A such that X_A FORS Y.

**Example 1.** Suppose that the return distributions of markets A and B are jointly elliptically distributed. Suppose the markets have the same number of assets n, vectors of averages Q_B is negative semidefinite. Then market A strongly dominates market B with respect to the increasing concave multivariate order (see [1]). Moreover market A weakly dominates market B with respect to the concave order since portfolio x'μ_A ≥ x'μ_B and x'Q_B x ≥ x'Q_A x for any vector x ≥ 0. Observe that the weak dominance between the markets is also known in ordering as the increasing positive linear concave multivariate order (see [1]).

Example 1 can be used in financial applications. In particular, if we assume that the returns of different markets are jointly elliptically distributed and they are uniquely determined by a risk measure and a reward measure, we can order the markets in a reward-risk framework. On the other hand, if we assume that the distribution does not have finite variance, the mean-variance approach is not appropriate. In this paper we propose a sub-Gaussian distributional assumption, which is quite more suitable for dealing with financial problems (see [7] and the references therein). In particular, we denote the univariate Pareto-Lévy stable distribution by S_x(σ, β, μ), where α ∈ (0,2) is the so-called stability index, which specifies the asymptotic behavior of the tails, σ > 0 is the dispersion parameter, β ∈ [−1,1] is the skewness parameter and μ ∈ ℝ is the location parameter. We consider the same notion used in [8].

A quite easy way to deal with stable distributions is to assume that the vector of returns follows a sub-Gaussian distribution. All components of a sub-Gaussian distribution are α-stable distributions, obtained by setting the skewness parameter β = 0, i.e. they are symmetric α-stable distributions. Thus, we propose to base the comparison between markets on: i) the vectors of averages; ii) the matrices of dispersion and iii) the stability indices. The motivation is that empirical evidence leads us to strongly suspect that, in the univariate case, if a distribution with heavier tails cannot dominate, at the second order (SSD), a distribution with higher expectation, inferior dispersion but heavier tails. Figure 1 and Figure 2 actually show that, on fixed values of σ, μ, the distribution with heavier tails is dominated by the distribution with lighter tails.

This concept can be applied in a multivariate context, generalizing the multivariate mean-variance approach described in Example 1, by taking into account the asymptotic behavior of the tail distributions. This yields the following definition of asymmetric multivariate dominance among financial markets.

**Definition 4.** Assume that the markets A and B have an equal number of assets n. Assume that the markets A and B are stable sub-Gaussian distributed with stability indices α_A and α_B, vectors of averages μ_A and μ_B, and dispersion matrices Q_A and Q_B. We say that market/sector A dominates market/sector B with respect to the asymptotic increasing concave multivariate order if α_A > α_B, μ_A ≥ μ_B and (Q_A − Q_B) is negative semidefinite. We say that market/sector A weakly dominates market/sector B with respect to the asymptotic increasing concave multivariate order if α_A > α_B, and, for any vector x ≥ 0, x'μ_A ≥ x'μ_B and x'Q_A x ≤ x'Q_B x.
In the following empirical analysis definition 4 to determine in practice if there exists the asymptotic increasing concave weak dominance among some equity markets.

### III. DOMINANCE COMPARISON AMONG THE US, CHINESE, JAPANESE AND GERMAN STOCK MARKETS

In this section, we evaluate the weak asymptotic multivariate dominance among the US, Chinese, Japanese and German stock markets. In particular, we consider the stocks of: NYSE and NASDAQ (US); Shanghai, Shenzhen and Honk Kong stock exchanges (China); Tokyo, Nagoya and Osaka stock exchanges (Japan); Frankfurt and Berlin stock exchanges (Germany).

First of all, we examine the statistical characteristics of the returns of each market. Then, we verify the dominance among stock markets during the decade 2004–2014. Since, in practical contexts, it is not easy to obtain the strong stochastic dominance among markets, then we verify if the conditions for the asymptotic weak dominance hold, under the implicit assumption the vector of returns of each market is i) normally distributed; ii) alpha stable sub-Gaussian distributed.

**i)** We assume that the returns of each country follows a Gaussian distribution with vector of means $\mu_A$ and variance-covariance matrix $Q_A$. For each couple of countries, we determine the so called alpha-mean-dispersion efficient frontier, computing the portfolio with minimum dispersion $x'Q_Ax$, for any fixed mean $x'\mu_A$, and finally we compare the efficient frontiers verifying if the conditions $\alpha_A > \alpha_B$, $x'\mu_A \geq x'\mu_B$ and $x'Q_Ax \leq x'Q_Bx$ hold.

The results of the two approaches are summarized in the Table 1.

<table>
<thead>
<tr>
<th>Tab. 1 Number of trimesters (January 2004- December 2014) when dominance among markets holds.</th>
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<td><strong>Mean-Variance comparison</strong></td>
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<tr>
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Table 1 reports the number of times (trimesters) when a market dominates another, in terms of reward-risk analysis, during the decade January 2004–December 2014. First of all, we observe that there exists a strong difference between the comparison based on mean-variance efficiency and the alpha-mean-dispersion efficiency. We observe that US and Chinese markets dominate the other two more frequently in the mean-variance framework. On the other hand, using the alpha-mean-dispersion criterion, only the Chinese market dominates few times the German and US markets. Moreover, we observe that the Japanese market never dominates the others and it is often dominated in terms of mean-variance. However, Japanese market is never dominated in terms of alpha-mean-dispersion efficient frontier, because it presents lower kurtosis and smaller tails, as also observed in Table 1. Therefore, from this analysis, the most performing market is the Chinese emerging market.

**IV. CONCLUSION**

We introduced a methodology aimed at comparing different financial markets/sectors from the point of view of a non-satiable risk-averse investor, the method is applied to four stock markets (US, German, Japan and China). The method could be very useful for investors who want to optimize their international portfolio, in particular, this analysis can be generally applied to preselect the “best” markets to invest in.

In section 2, we proposed a definition of multivariate dominance among different markets and evaluate it with empirical comparison between markets, assuming that the returns are in i) normally distributed; ii) sub-Gaussian...
distributed. We observe that the mean-variance dominance (approach i) among different markets is verified several times, although we generally do not observe the asymptotic dominance (approach ii), except in few cases. In particular, while the Japanese stock market appears to be dominated in terms of mean-variance, it is never asymptotically dominated since it presents an index of stability generally higher compared to the other countries. This result suggests that the big losses observed during the crisis have a stronger impact in the US, China and German stock markets.

REFERENCES