Sensitivity analysis in optimal control of the Earth's climate system

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***Abstract*—**In this paper, we, in general terms, formulate the optimal control problem for the Earth climate system, and discuss sensitivity analysis methods applicable to estimate the impact of geoengineering (climate engineering) technologies on the climate system of our planet.

***Keywords*—**Climate change, climate engineering, optimal control, sensitivity analysis.

I. Introduction

Observed global climate change represents, apparently, one of the most significant environmental, social and economic challenges for the humanity in the XXI century. According to climate theory, and observational data and research findings, the global warming is a man-made phenomenon caused by the increase of concentrations of greenhouse gases (GHGs) such as carbon dioxide (CO_2) and methane (CH_4). Observations show that since the beginning of the 20-th century, the Earth's global average surface temperature has increased by almost 0.8 °C, with about two-thirds of the increase occurring since 1980. Presently, global warming is acknowledged by both the scientific community and majority of policymakers. The IPCC Fifth Assessment Report (AR5) [1] provides a distinct view of the up-to-date state of scientific knowledge regarding climate change. It is recognized, that mankind is causing global warming by anthropogenic CO_2 emissions generated by human activities through combustion of fossil fuel, mainly coal, oil, natural gas and wood. Due to the burning of fossil fuels and destruction of native forests, the concentration of carbon dioxide is increased from 280 to more than 400 parts per million (ppm) since the beginning of the so-called Industrial Revolution (~1750).

The most appropriate solution to reduce global warming is cutting down the anthropogenic emissions of GHGs. However, this is hardly achievable because the world economic growth and increasing population require more and more energy resources generating more and more GHG emissions. Carbon dioxide-free renewable energy resources and energy efficiency measures at the moment are not the alternative because they are very expensive and require long time to achieve tangible results. Realizing that, scientific community proposed several solutions, known as geoengineering, to stabilize the global climate (e.g. [2]-[9]). In general, geoengineering is divided on two main categories: carbon dioxide removal technologies, and solar radiation management. A number of geoengineering solutions are offered to date, however all of them introduce uncertainties and unexpected consequences that must be explored.

Climate engineering is purposeful process, i.e. the process having a special purpose and, therefore, outcome, which can be formulated in various ways and should be achieved somehow. In this regards, climate engineering is, in essence, the process of controlling the climate system that can be examined from the standpoint of control theory [10]-[14]. Within the framework of control theory, climate system is considered as a self-regulating feedback cybernetic system, in which the climate system itself represents control object, and the role of controller is given to human operators. The Earth's climate system (ECS) is a complex, interactive, nonlinear dynamical system consisting of the atmosphere, hydrosphere, cryosphere, lithosphere and biosphere. The state of the ECS at a given time and place with respect to variables such as temperature, barometric pressure, wind velocity, moisture, precipitations is known as the weather. Climate is usually defined as "average weather" or, in other words, as an ensemble of states traversed by the climate system over a sufficiently long period of time. Commonly, this period corresponds to ~30 years, as defined by the World Meteorological Organization. Since the ECS is a unique physical object with a large number of specific features [15]-[18], control of physical and dynamical process occurring in the ECS is an extremely complex and difficult problem. Synthesis of control systems of such natural objects represents a multidisciplinary research area which is developed on the ideas and methods of optimal control theory, dynamical systems theory, technical cybernetics, climate physics and dynamics, and other academic disciplines.

It is clear that the success of climate engineering strongly depends on the understanding of physics, chemistry and dynamics of climate processes as well as the availability of enabling technologies. However, we also need to know the ECS response to geoengineering. The estimation of the effectiveness of climate engineering approaches and the assessment of their impact on the climate system can be performed by the method of mathematical/numerical modeling

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since the ability of laboratory simulations of the ECS are, with rare exceptions, very limited. In general, synthesis of optimal control systems is based on a performance measure to be optimized and a mathematical model of the dynamical system to be controlled. It is very important that the performance of optimally controlled process depends on the accuracy, reliability and adequacy of the model used. Mathematical models of the ECS used in variety of applications are derived from a set of multidimensional nonlinear differential equations in partial derivatives, which are the equations of fluid dynamics and thermodynamics that describe dynamical, physical and chemical processes in the ECS. Thus, the ECS refers to a class of the so-called distributed parameter systems (e.g. [19]-[21]). Note that ECS models are mostly deterministic with a large phase space dimension [22]-[24]. Apparently, due to their complexity, equations that describe the evolution of the ECS cannot be solved analytically with an arbitrary set of initial conditions, but only numerically using various types of finite-dimensional approximations such, for example, as Galerkin projection or finite-difference methods.

In mathematical models, large-scale intervention in the ECS can be described parametrically, or, in other words, effects of geoengineering activities can be parameterized using conventional parameterization schemes of sub-grid physical processes, first of all schemes that describe radiative processes (short-wave solar radiation, long-wave emissions of the Earth) in the ECS. Thus, the impact of geoengineering methods can be estimated by studying the sensitivity of the ECS with respect to parameters that reflect the influence of external forcing on physical processes occurring in the ECS [25]-[28]. However, dynamical systems used to model the ECS, essentially nonlinear and under certain conditions they exhibit aperiodic oscillations, which are known as the phenomenon of deterministic chaos [29]. For such systems, the conventional methods of sensitivity analysis are not sufficiently effective, since calculated sensitivity functions are uninformative and inconclusive [30]-[33]. In this context, the exploration of sensitivity of nonlinear models of the ECS with respect to their parameters and external forcing require special consideration.

In this paper, we, in general terms, formulate the optimal control problem for the ECS and discuss sensitivity analysis approaches applicable to estimate the impact of geoengineering methods on the ECS.

II. Sensitivity Analysis of Climate Models

A. Formal Model of the Earth Climate System

Let us consider the continuous dynamical system, which formally describes the evolution of the ECS in the bounded space-time domain $\Omega_t = \Omega \times [0,\tau]$, where Ω is the space domain representing the earth's sphere and $[0,\tau]$ is the time interval:

$$\frac{\partial \varphi(r,t)}{\partial t} = L(\varphi(r,t), \lambda(r,t)),\ \varphi(r,0) = \varphi_0 . \quad (1)$$

Here $\varphi \in Q(\Omega_t)$ is the state vector of a system, where $Q(\Omega_t)$ is the infinite real space of sufficiently smooth state functions satisfying some problem-specific boundary conditions at the boundary $\partial\Omega$ of the domain Ω; $r \in \Omega \subset \mathrm{R}^3$ is a vector of spatial variables; $t \in [0,\tau]$ is the time; L is a nonlinear operator that describes the dynamics, dissipation and external forcing of the system; $\lambda \in G(\Omega_t)$ is the system parameter vector, where $G(\Omega_t)$ is the domain of admissible values of the parameters; and φ_0 is the initial state estimate. In order to obtain a system with a finite number of degrees of freedom the equations (1) can be projected onto the subspace spanned by the orthogonal basis $\{\psi_i\}$ so that

$$\varphi(r,t) \approx \sum_{i=1}^{n} x_i(t)\psi(r) . \quad (2)$$

Substituting (2) into (1) and using then the Galerkin method, we can obtain the dynamical system that is described by the set of ordinary differential equations (ODEs):

$$\dot{x}(t) = f(x,\alpha),\ t \in [0,\tau 0],\ x(0) = x_0 , \quad (3)$$

where $x \in X \subseteq R^n$ is the state vector, $\alpha \in P \subseteq R^m$ is the vector of parameters, $f \in R^n$ is a nonlinear vector-function defined in the domain $X \times P \times [0,\tau]$, and x_0 is a given initial conditions. The finite-dimensional dynamical system (3) can be obtained by discretization in space of the equations (1).

Given the system state x_0 at time $t = 0$, we define the trajectory of x_0 under f to be the sequence of points $\{x_k\}$, $k \in Z_+$, such that $x_k = f^k(x_0)$, where f^k indicates the k-fold composition with itself, and $f^0(x) \equiv x_0$. Thus, given the vector-function f and the initial conditions x_0, equation (3) uniquely specifies the orbit of a dynamical system. For the ECS, the vector-function f is nonlinear. This nonlinearity arises from numerous feedbacks existed in a broad spectrum of oscillations and external forcing caused by natural and anthropogenic processes. Furthermore, nonlinear motions in the atmosphere, which is the most fast-oscillating component of the ECS, under certain conditions demonstrate a chaotic behavior. The ECS is a dissipative dynamical system, which possess a global attractor. This implies that there exists an absorbing set which is bounded set is the phase space that attracts any trajectory of the system. In other words, the norm of the solution of the model equations with arbitrary initial conditions, from a certain moment of time t^*, does not exceed some fixed value: $\|x(t)\| < c_0, t \ge t^*$. Usually, in many applications, the ECS evolution is considered on its attractor assuming that the system is ergodic.

Since the right-hand sides of climate system equations (3) include a large number of various parameters, the problem occurs of the system stability with respect to changes in the parameters. This problem is closely related to the parametric sensitivity analysis of dynamical systems.

B. Conventional Methods of Sensitivity Analysis

To analyze the sensitivity of the system with respect to parameter variations let us introduce a response function:

$$J(x,\alpha)=\int_0^{\tau} F(t;x,\alpha)dt\ , \tag{4}$$

where F is a (nonlinear) function of state variables x and parameters α. Let α^0 be the unperturbed parameter vector, and x^0 be the state vector which is obtained by solving the equation (1) with $\alpha=\alpha^0$. The impact of parameter variations on the system performance is quantified by the gradient of the response function with respect to α around the unperturbed point (x^0,α^0):

$$\nabla_{\alpha} J\left(x^0,\alpha^0\right)=\left(\frac{dJ}{d\alpha_1},\ldots,\frac{dJ}{d\alpha_m}\right)^{\mathrm{T}}\Bigg|_{x^0,\alpha^0}. \tag{5}$$

Particularly, the influence of parameter α_j is calculated as

$$\frac{dJ}{d\alpha_j}=\sum_{i=1}^{n}\frac{\partial J}{\partial x_i}\frac{\partial x_i}{\partial \alpha_j}+\frac{\partial J}{\partial \alpha_j}=\sum_{i=1}^{n}S_{ij}\frac{\partial J}{\partial x_i}+\frac{\partial J}{\partial \alpha_j}\ ,$$

where $S_{ij}\equiv \partial x_i/\partial\alpha_j$ are the sensitivity coefficients [28]:

$$S_{ij}=\lim_{\delta\alpha_j\to 0}\left[\frac{x_i\left(\alpha_j+\delta\alpha_j\right)-x_i\left(\alpha_j\right)}{\delta\alpha_j}\right]$$

The first order sensitivity estimate for variations in the parameter α_j is given by

$$\frac{dJ(x,\alpha)}{d\alpha_j}\bigg|_{x^0,\alpha_j^0}\approx\frac{J\left(x^0+\delta x;\alpha_1^0,\ldots,\alpha_j^0+\delta\alpha_j,\ldots,\alpha_m^0\right)-J\left(x^0,\alpha^0\right)}{\delta\alpha_j}.$$

This equation approximates the derivative of the first order; therefore the accuracy of approximation essentially depends on the choice of the parameter variation $\delta\alpha_j$. Generally, this selection is made arbitrarily bearing in mind that the value of $\delta\alpha_j$ is bounded below by the round-off error. Introducing the Gâteaux differential [26], the sensitivity analysis problem can be considered in the differential formulation that eliminates the need to set the value of $\delta\alpha$. The Gâteaux differential is defined as

$$\delta J\left(x^0,\alpha^0\right)=\int_0^{\tau}\left(\frac{\partial F}{\partial x}\bigg|_{x^0,\alpha^0}\cdot\delta x+\frac{\partial F}{\partial\alpha}\bigg|_{x^0,\alpha^0}\cdot\delta\alpha\right)dt\ , \tag{6}$$

where δx is the variation in the state vector due to the variation in the parameter vector in the direction $\delta\alpha$. Linearizing the model (3) around the unperturbed trajectory $x^0(t)$, we obtain the following system of variational equations for calculating $\delta\alpha$:

$$\frac{\partial\delta x}{\partial t}=\frac{\partial f}{\partial x}\bigg|_{x^0,\alpha^0}\cdot\delta x+\frac{\partial f}{\partial\alpha}\bigg|_{x^0,\alpha^0}\cdot\delta\alpha,\ t\in[0,\tau],\ \delta x(0)=\delta x_0\ . \tag{7}$$

The model (6) is known as a tangent linear model. Variations δx obtained from the equation (7) are then used in the equation (6) for evaluating the Gâteaux differential. Since $\delta J\left(x^0,\alpha^0\right)=\left\langle\nabla_\alpha J,\delta\alpha\right\rangle$, where $\langle\cdot,\cdot\rangle$ is a scalar product, then the model sensitivity with respect to the variations in the parameters can be estimated by calculating the components of the gradient $\nabla_\alpha J$. However, this "one-at-a time" method, in spite of its simplicity, requires significant computational resources if the number of model parameters is large. The use of adjoint equations allows us obtaining the required sensitivity estimates within a single computational experiment, since the gradient can be calculated as [26]:

$$\nabla_{\alpha} J\left(x^0,\alpha^0\right)=\int_0^{\tau}\left[\frac{\partial F}{\partial\alpha}\bigg|_{x^0,\alpha^0}-\left(\frac{\partial f}{\partial\alpha}\bigg|_{x^0,\alpha^0}\right)^{\mathrm{T}}\cdot x^*\right]dt\ , \tag{8}$$

where the vector-valued function x^* is the solution of the adjoint model:

$$-\frac{\partial x^*}{\partial t}-\left(\frac{\partial f}{\partial x}\bigg|_{x^0,\alpha^0}\right)^{\mathrm{T}}\cdot x^*=-\frac{\partial F}{\partial x}\bigg|_{x^0,\alpha^0},\ t\in[0,\tau],\ x^*(\tau)=0\ . \tag{9}$$

The adjoint equations (9) are integrated backward in time.

As discussed in [28], general solutions of sensitivity equations for oscillatory nonlinear dynamical systems grow unbounded as time tends to infinity; therefore, sensitivity functions calculated by conventional approaches have a high degree of uncertainty. The reason is that nonlinear dynamical systems that exhibit chaotic behavior are very sensitive to its initial conditions. Thus, the solutions to the linearized Cauchy problem (3) grow exponentially $\|\delta x(t)\|\approx\|\delta x(0)\|e^{\lambda t}$, where $\lambda>0$ is the leading Lyapunov exponent. As a result, calculated sensitivity functions (coefficients) include a fairly large error, becoming uninformative and inconclusive [30-33].

C. Fluctuation-Dissipation Theorem

To estimate the ensemble-averaged response of the ECS to small external forcing Leith [34] has proposed using the fluctuation-dissipation theorem (FDT). According to the FDT, under certain assumptions, the response of stochastic dynamical system to infinitesimal external perturbations is described by the covariance matrix of the unperturbed system:

$$\left\langle\delta x(t)\right\rangle=\int_0^t C(\tau)C^{-1}(0)d\tau\cdot\delta\alpha^{ext}\ , \tag{10}$$

where $\delta\alpha^{ext}$ is an external forcing, $\langle\cdot\rangle$ is the symbol means an ensemble average over realizations, and $C(\tau)$ is a τ–lagged covariance matrix of x. It is usually assumed that the system is close to thermal equilibrium and the probability density function of the unforced system is Gaussian. However, the climate system is characterized by a strong external forcing and dissipation, making it a system for which the standard assumptions of equilibrium statistical mechanics do not hold.

D. Sensitivity Analysis Based on Shadowing Property

In climate studies, the average values of sensitivity functions $\nabla_\alpha\left\langle J(\alpha)\right\rangle$ over a certain period of time are usually considered as one of the most important measures of sensitivity, where J is a generic objective function (4). However, the gradient of J with respect to α cannot be correctly estimated within the framework of conventional methods of sensitivity analysis since for chaotic systems it is observed that [30-33]

$$\nabla_\alpha\left\langle J(\alpha)\right\rangle\neq\left\langle\nabla_\alpha J(\alpha)\right\rangle.$$

This is because the integral

$$\mathcal{I} = \lim_{\tau\to\infty}\int_0^\tau \lim_{\delta\alpha\to 0}\left[\frac{J(\alpha+\delta\alpha)-J(\alpha)}{\delta\alpha}\right]dt \qquad (11)$$

does not possess uniform convergence and two limits ($\tau\to\infty$ и $\delta\alpha\to 0$) would not commute. The “shadowing” approach for estimating the system sensitivity to variations in its parameters suggested in [31] and [32] allows us to calculate correctly the average sensitivities $\langle\nabla_\alpha J(\alpha)\rangle$ and therefore to make a clear conclusion with respect to the system sensitivity to its parameters. This approach is based on the theory of pseudoorbit shadowing in dynamical systems [35], [36], which is one of the most rapidly developing components of the global theory of dynamical systems and classical theory of structural stability [37].

Naturally, pseudo- (or approximate-) trajectories arise due to the presence of round-off errors, method errors, and other errors in computer simulation of dynamical systems. Consequently, we will not get an exact trajectory of a system, but we can come very close to an exact solution and the resulting approximate solution will be a pseudotrajectory. The shadowing property (or pseudo orbit tracing property) means that, near an approximate trajectory, there exists the exact trajectory of the system considered, such that it lies uniformly close to a pseudotrajectory. The shadowing theory is well-developed for the hyperbolic dynamics, which is characterized by the presence of expanding and contracting directions for derivatives. The study of shadowing problem was originated by D.V. Anosov [38] and R. Bowen [39].

Let (*M*, dist) be a compact metric space and let $f: M\to M$ be a homeomorphism (a discrete dynamical system on *M*). A set of points $X=\{x_k : k\in\mathbb{Z}\}$ is a *d*-pseudotrajectory $(d>0)$ of f if

$$\mathrm{dist}\left(x_{k+1}, f(x_k)\right) < d,\ \ k\in\mathbb{Z}\ .$$

Here the notation $\mathrm{dist}(\ ,\ )$ denotes the distance in the phase space between two geometric objects within the brackets.

We say that f has the shadowing property if given $\varepsilon>0$ there is $d>0$ such that for any *d*-pseudotrajectory $X=\{x_k : k\in\mathbb{Z}\}$ there exists a corresponding trajectory $Y=\{y_k : k\in\mathbb{Z}\}$, which ε -traces *X*, i.e.

$$\mathrm{dist}\left(x_k, y_k\right) < \varepsilon,\ \ k\in\mathbb{Z}\ .$$

The shadowing lemma for discrete dynamical systems [35] states that for each $\varepsilon>0$, there exists $d>0$ such that each d-pseudotrajectory can be ε-shadowed. The definition of pseudotrajectory and shadowing lemma for flows (continuous dynamical systems) are more complicated than for discrete dynamical systems [35]. Let $\Phi^t:\mathbb{R}\times M\to M$ be a flow of a vector field *X* on *M*. A function $g:\mathbb{R}\to M$ is a d-pseudotrajectory of the dynamical system Φ^t if the inequalities

$$\mathrm{dist}\left(\Phi^t\left(t, g(\tau)\right), g(\tau+t)\right) < d$$

hold for any $t\in[-1,1]$ and $\tau\in\mathbb{R}$. The “continuous” shadowing lemma ensures that for the vector field *X* generating the flow Φ^t, the shadowing property holds in a small neighborhood of a compact hyperbolic set for dynamical system Φ^t. However, the shadowing problem for continuous dynamical systems requires reparameterization of shadowing trajectories. This is the case because for continuous dynamical systems close points of pseudotrajectory and true trajectory do not correspond to the same moments of time. A monotonically increasing homeomorphism $h:\mathbb{R}\to\mathbb{R}$ such that $h(0)=0$ is called a reparameterization and denoted by *Rep*. For $\varepsilon>0$, *Rep*(ε) is defined as follows [35]:

$$Rep(\varepsilon)=\left\{h\in Rep : \left|\frac{h(t_1)-h(t_2)}{t_1-t_2}-1\right|\le\varepsilon\right\} \text{ for any different } t_1,t_1\in\mathbb{R}\ .$$

To illustrate the applicability of this method, let us consider the continuous one parameter dynamical system $\dot{x}=f(x,\alpha)$ on the time interval $[0,\tau]$. The sensitivity analysis aims to estimate the sensitivity coefficient $S_\alpha=\partial x/\partial\alpha$. Let $x'(t)$ be the pseudo-orbit obtained by integration of the system equations with perturbed parameter $\alpha'=\alpha+\delta\alpha$, where $\delta\alpha$ is the variation in α. Since the pseudotrajectory $x'(t)$ stays uniformly close to the “true” orbit $x(t)$ obtained with unperturbed parameter α, the integral (11) is convergent and the average sensitivities $\langle\nabla_\alpha J(\alpha)\rangle$ can be easily estimated. Let us introduce the following transform $x'(x)=x+\delta x(x)$. It can be shown that $\delta f(x)=A\delta x(x)$, where $A=\left[-(\partial f/\partial x)+(d/dt)\right]$ is a “shadow” operator. Thus, to find a pseudo-orbit we need to solve the equation $\delta x=A^{-1}\delta f$, i.e. we must numerically invert the operator A for a given δf . To solve this problem, we decompose functions δx and δf into their constituent Lyapunov covariant vectors $\upsilon_1(x),\ldots,\upsilon_n(x)$:

$$\delta x(x)=\sum_{i=1}^{n}\psi_i(x)\upsilon_i(x)\,,\quad \delta f(x)=\sum_{i=1}^{n}\varphi_i(x)\upsilon_i(x)\,,$$

and then compute the Lyapunov exponents λ_i and vectors $\upsilon_1(x),\ldots,\upsilon_n(x)$. By executing the spectrum decomposition of δf along the trajectory $x(t)$ we can obtain $\varphi_i(x)$, $i=1,...,n$ and then calculate the expansion coefficients $\psi_i(x)$, $i=1,\ldots,n$ using the equations

$$\frac{d\psi_i(x)}{dt}=\varphi_i(x)+\lambda_i\psi_i(x),\ i=1,\ldots,n\ ,$$

which are derived from the dynamical system equations. The expansion coefficients $\psi_i(x)$ are used to compute δx along the trajectory. By averaging δx over the time interval $[0,\tau]$ we can obtain the desired sensitivity estimate $S_\alpha=\langle\delta x\rangle/\delta\alpha$.

III. Optimal Control Problem Statement

Let the controllable climate system on the time interval $t\in[0,\tau]\subset\mathbb{R}^+$ is described by the following set of ordinary differential equations

$$\dot{x} = f(x,u),\ x(0) = x_0, \quad (12)$$

where $u \in U \subset \mathbb{R}^m$ is a control vector. Note that in (12) uncontrolled parameters are omitted. We assume that control parameters belong to a set of admissible controls $u \in \mathcal{U} \subset U$ and depend on the system state, i.e. $u(t) = g(t, x(t))$, which implies that equations (12) describe a closed-loop control system, representing the ECS. The set $\mathcal{U}$ is defined on the basis of physical and technical feasibility taking into account the specific properties of the ECS as a control object. Under certain conditions [40] the Cauchy problem (12) has a unique solution defined on a time interval in $\mathbb{R}^+$. However, we cannot a priori determine whether the ECS is controllable or not. Conclusion concerning controllability of the system can only be made by solving a specific problem.

The main objective of the problem considered is to synthesize the control law that ensures the achievement of the desired results. Since these results are expressed in terms of extremal problem, we are specifically interested in synthesis of an optimal control. Stabilization of the ECS around the reference phase space trajectory in order to weaken the global warming represents one of the most important problems relevant to the optimal control of climate processes. For this particular class of problems, the differential equations (12) are linearized with respect to the natural (reference) trajectory $x^0(t)$ caused by external natural unperturbed forcing $u^0(t)$:

$$\frac{d\delta x(t)}{dt} = \left.\frac{\partial f}{\partial x}\right|_{x^0,u^0} \cdot \delta x(t) + \left.\frac{\partial f}{\partial u}\right|_{x^0,u^0} \cdot \delta u(t),\ \ \delta x(0) = 0, \quad (13)$$

where δx is the perturbation of natural orbit of the ECS due to anthropogenic disturbances, δu is a control vector to ensure the stabilization of the ECS' trajectory, $\partial f/\partial x$ and $\partial f/\partial u$ are the Jacobian matrices. Naturally, we have to assume that

$$u = u^0 + \delta u,\ \ |\delta u| << |u^0|;\ x = x^0 + \delta x,\ \ |\delta x| << |x^0|.$$

The optimal control problem is formulated as follows:

Find the control vector

$$\delta u^*(t) \in \mathcal{U} \quad (14)$$

generating the correction of the natural orbit

$$\delta x^*,\ \ x^0 + \delta x^* \in \mathcal{X} \subset X \quad (15)$$

such that the performance index J is minimized:

$$\delta u^* = \arg\min_{\delta u \in \mathcal{U}} J(\delta x, \delta u), \quad (16)$$

$$J = \frac{1}{2}\delta x^{\mathrm{T}}(\tau) G \delta x(\tau) + \frac{1}{2}\int_0^\tau \left[\delta x^{\mathrm{T}}(t) W \delta x(t) + \delta u^{\mathrm{T}}(t) Q \delta u(t)\right] dt \quad (17)$$

where $W(t)$ and G are weighting positive semi-definite $n \times n$ matrices, normalizing the energy of the ECS per unit mass, $Q(t)$ is a weighting positive define $m \times m$ matrix, normalizing the energy of control actions per unit mass.

The stabilization problem is solved, given the fact that the system travels along its natural trajectory that is subject to external natural forcing. The control objective is to keep $\delta x(t)$ close to zero using control actions $\delta u(t)$. The information on the ECS state $x(t)$ is obtained by measurement devices and instruments followed by the processing using data assimilation procedure. The problem (14)-(16) includes a set X at which the functional J is defined, and constraints on the model state given by the subset $\mathcal{X}$ of a set X. The dynamic constraints are given by equations (13). There are several methods available for solving the problem (14)-(16): classical methods of the variational calculus, dynamical programming, the Pontryagin's maximum principle and other methods.

It is important to underline that the formulation of performance index (17) depends on the problem under consideration and there are no universal approaches how it can be specified.

IV. Application of sensitivity methods to the low order climate modeling system

In order to explore the applicability of geoengineering methods and technologies to climate manipulation, we have to choose the system parameters that can be considered as control variables. Then, sensitivity analysis, discussed in Section II, allows us to establish the reaction of the dynamical system onto changes in these control variables and to define their range of values. A wide spectrum of climate models of various complexity is used in simulation of the ECS. The exploration of the ECS requires considerable computational resources. For simple enough low dimensional models, the computational cost is minor and, for that reason, models of this class are widely applied as simple test instruments to emulate more complex systems such as the ECS. In this paper, to simulate the ECS we will use the following low order coupled nonlinear dynamical system, which is composed of fast (the "atmosphere") and slow (the "ocean") subsystems [41]:

$$\dot{x} = -y^2 - z^2 - ax + aF, \quad (18)$$

$$\dot{y} = xy - cy - bxz + G + \alpha X, \quad (19)$$

$$\dot{z} = xz - cz + bxy + \alpha Y, \quad (20)$$

$$\dot{X} = -\omega Y - \beta y, \quad (21)$$

$$\dot{Y} = \omega X - \beta z, \quad (22)$$

where x is the intensity of the symmetric, globally averaged westerly wind current (equivalent to the meridional temperature gradient); y and z are the amplitudes of cosine and sine phases of a series of superposed large scale eddies, which transport heat poleward; F and G represent the thermal forcing terms due to the average north-south temperature contrast and the earth-sea temperature contrast, respectively. The term b represents displacement of the waves due to interaction with the westerly wind. The coefficient a, if less than 1, allows the westerly wind current to damp less rapidly than the waves. The time unit of t is estimated to be ten days.

This low order coupled system allows us to mimic the atmosphere-ocean system and therefore may serve as a key

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element of a theoretical and computational framework for the study of various aspects of ECS including geoengineering. Note that the atmospheric system described by equations (18)-(20) represents a chaotic Lorenz system [42], while the ocean system is a simple harmonic oscillator.

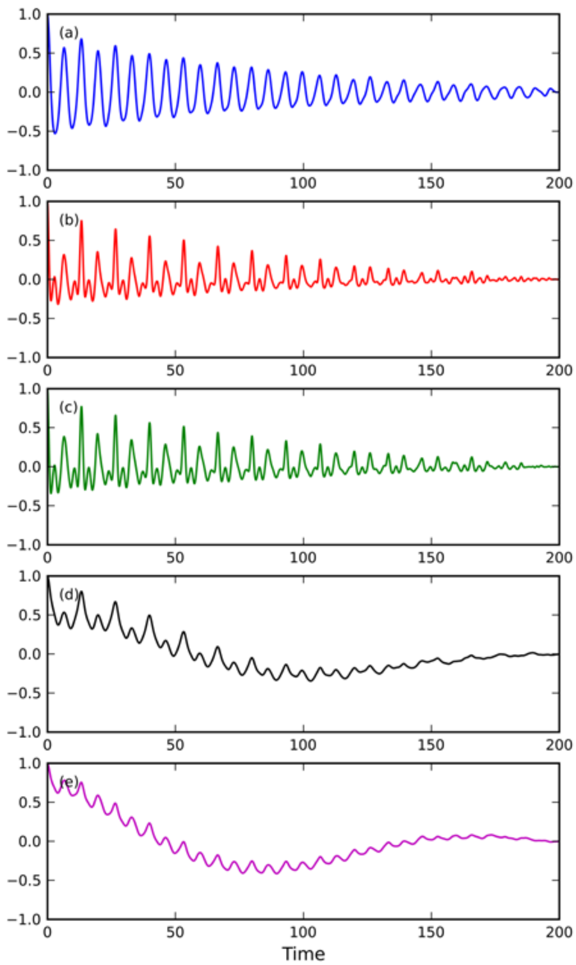

Fig. 1 Autocorrelation functions for variables x (a), y (b), z (c), X (d) and Y (e)

Let us consider some results of numerical experiments. Sensitivity theory shows that general solutions of sensitivity equations for oscillatory nonlinear dynamical systems grow unbounded as time tends to infinity; therefore, sensitivity functions calculated by conventional approaches have a high degree of uncertainty. The reason is that nonlinear dynamical systems that exhibit chaotic behavior are very sensitive to its initial conditions. Thus, the solutions to the linearized Cauchy problem (3) grow exponentially as $\|\delta x(t)\| \approx \|\delta x(0)\| e^{\lambda t}$, where $\lambda > 0$ is the leading Lyapunov exponent. As a result, calculated sensitivity coefficients contain a fairly large error. Application of conventional sensitivity analysis methods to the system (18)-(22) confirms this point: envelopes of calculated sensitivity functions (coefficients) grow in time while sensitivity coefficients themselves exhibit oscillating behavior. Thus, obtained sensitivity coefficients are inherently uninformative and misleading, and we cannot make a clear conclusion from them about system sensitivity to variations in the model parameters.

The FDT also cannot provide clear information about the system sensitivity with respect to its parameters. In Fig. 1 autocorrelation functions (ACFs) are presented for realizations of all dynamic variables of the model (18)-(22). Using ACFs we can easily calculate the system response functions. However, for oscillatory ACFs calculated response functions are uninformative.

Using the shadowing method allows us to calculate the average sensitivity functions (coefficients) that can be easily interpreted. However:

(1) The shadowing property of dynamical systems is a fundamental feature of hyperbolic systems, but most physical systems are non-hyperbolic. Despite the fact that much of shadowing theory has been developed for hyperbolic systems, there is evidence that non-hyperbolic attractors also have the shadowing property. In theory this property should be verified for each particular dynamical system, but this is more easily said than done.

(2) The applicability of the shadowing method for sensitivity analysis of modern climate models is a rather complicated problem since these models are quite complex and they contain numerous input parameters. Thus, further research and computational experiments are required. We are confident that by using the basic ideas of the shadowing method, it is possible to better understand the sensitivity analysis of climate models of various levels of complexity.

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