Analytical solution of a problem on MHD flow in a rectangular duct

Elena Ligere, Ilona Dzenite and Aleksandrs Matvejevs

Abstract — This paper presents an analytical solution of the MHD problem on a fully developed flow of a conducting fluid in a duct with the rectangular cross-section, located in a uniform external magnetic field, and under a slip boundary condition on side walls of the duct. The flow is driven by a constant pressure gradient. The case of perfectly conducting Hartmann walls and insulating side walls is considered. The solution is derived by using integral transforms.

Keywords — Integral transforms, magnetohydrodynamic duct flow, slip boundary condition.

I. Introduction

A FLOW of a conducting fluid in the presence of external magnetic field produces a variety of new effects, studied by magnetohydrodynamics (MHD), the discipline combining the classical fluid mechanics and electrodynamics. The MHD effects are widely exploited both in technical devices (e.g., in pumps, flow meters, generators) and industrial processes in metallurgy, material processing, chemical industry, industrial power engineering and nuclear engineering. Channels, in particular rectangular and circular channels, are common parts of many MHD devices. Therefore, investigation of MHD phenomena in channels with conducting fluids is quite important.

The motion of conducting fluid in external magnetic field is described by the system of MHD equations, containing Navier-Stokes equation for the motion of incompressible viscous fluid with the additional term corresponding to the Lorentz force and Maxwell's equations (see [1]). In MHD the number of exact solutions, obtained analytically, is limited due to the nonlinearity of the Navier-Stokes equation. The exact solutions have been obtained only for very specific problems; however, numerical methods are widely used for solving MHD problems.

The fully developed flows in rectangular ducts are well studied for different electric conductivities of the walls, but under "no slip" condition on the duct walls (for example, see [1]). Recently, in [2] three classic MHD problems are revisited on assuming a hydrodynamic slip condition at the interface between the electrically conducting fluid and the insulating walls. One of the problems studied analytically in [2] is the problem on a fully developed flow in the rectangular duct with insulating walls and a slip condition on the Hartmann walls (the walls perpendicular to the magnetic field).

This paper presents an analytical solution of the MHD problem on a fully developed flow of a conducting fluid in the duct with the rectangular cross-section, located in a uniform external magnetic field, and under a slip boundary condition on side walls of the duct. The obtained solution seems absent in literature.

The use of integral transforms or series expansion (see [3]) is one of the powerful method for obtaining analytical solutions of problems in mathematical physics. Also in MHD some problems with specific geometry of the flow and boundary conditions are well-solved by integral transforms (for example, see the author's works [4] - [7]).

The MHD problem of this paper is also solved by using integral transforms, but at first, the kernels of integral transforms has been derived and then used for solving the problem.

II. Problem Formulation

Consider the MHD problem on a fully developed flow of a conducting fluid in the rectangular duct with the perfectly conducting Hartmann walls at $z = \pm 1$ and non-conducting side walls at $y = \pm d$ (the walls parallel to the external magnetic field) with the slip boundary condition on the side walls (see Fig.1).

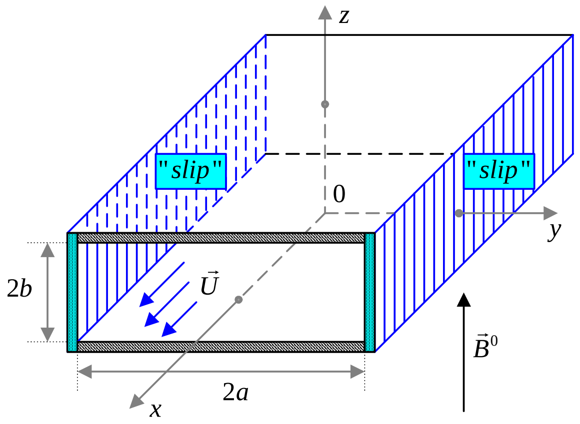

Fig. 1 MHD duct flow with a slip boundary condition

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The dimensionless MHD equations, describing the problem, have the form ([1], [2]):

$$\frac{\partial^2 U}{\partial z^2}+\frac{\partial^2 U}{\partial y^2}+1+Ha\frac{\partial b_x}{\partial z}=0\,, \tag{1}$$

$$\frac{\partial^2 b_x}{\partial z^2}+\frac{\partial^2 b_x}{\partial y^2}+Ha\frac{\partial U}{\partial z}=0\,, \tag{2}$$

where $\vec{V}=U(y,z)\cdot\vec{e}_x$ is the velocity of the fluid,

$\vec{b}=b_x(y,z)\cdot\vec{e}_x$ is the induced magnetic field,

$Ha=B_0h\sqrt{\sigma/\rho\nu}$ is the Hartmann number, which characterizes the ratio of electromagnetic force to viscous force; σ, ρ, ν are the conductivity, the density and the viscosity of the fluid, respectively.

The boundary conditions are

$$z=\pm1:\quad U=0\,,\quad \frac{\partial b_x}{\partial z}=0\,, \tag{3}$$

$$y=\pm d:\ U\pm\alpha\frac{\partial U}{\partial y}=0\,,\quad b_x=0\,, \tag{4}$$

where α is the slip length. The slip condition is given by the 3rd kind boundary condition ([2]).

III. Problem Solving

The problem is solved by using the integral transforms

$$\bar{u}(\lambda,y)=\int_{-1}^{1}U(y,z)K_1(\lambda,z)\,dz\,, \tag{5}$$

$$\bar{b}(\lambda,y)=\int_{-1}^{1}b_x(y,z)K_2(\lambda,z)\,dz\,, \tag{6}$$

where $K_1(\lambda,z)$ and $K_2(\lambda,z)$ are unknown kernels.

In order to find the unknown kernels, (1) is multiplied by $K_1(\lambda,z)$, (2) by $K_2(\lambda,z)$, and integrated with respect to z.

Thus, it yields

$$\frac{\partial U}{\partial z}K_1\Big|_{z=1}-\frac{\partial U}{\partial z}K_1\Big|_{z=-1}-UK_1'\big|_{z=1}+UK_1'\big|_{z=-1}+\int_{-1}^{1}UK_1''dz+$$
$$+\frac{d^2\bar{u}}{dy^2}+\int_{-1}^{1}K_1\,dz+Ha\left(b_xK_1\big|_{z=1}-b_xK_1\big|_{z=-1}-\int_{-1}^{1}b_xK_1'\,dz\right)=0\,, \tag{7}$$

$$\frac{\partial b_x}{\partial z}K_2\Big|_{z=1}-\frac{\partial b_x}{\partial z}K_2\Big|_{z=-1}-b_xK_2'\big|_{z=1}+b_xK_2'\big|_{z=-1}+$$
$$+\int_{-1}^{1}b_xK_2''\,dz+\frac{d^2\bar{b}}{dy^2}+Ha\left(UK_2\big|_{z=1}-UK_2\big|_{z=-1}-\int_{-1}^{1}UK_2'\,dz\right)=0 \tag{8}$$

Due to the boundary conditions (3), the following terms are equal to zero:

$$UK_1'\big|_{z=1}=UK_1'\big|_{z=-1}=0\,,\qquad \frac{\partial b_x}{\partial z}K_2\Big|_{z=1}=\frac{\partial b_x}{\partial z}K_2\Big|_{z=-1}=0\,,$$

$$UK_2\big|_{z=1}=UK_2\big|_{z=-1}=0\,. \tag{9}$$

The following additional conditions for the kernels are to be applied [3]:

$$K_1''(\lambda,z)=-\lambda^2K_1(\lambda,z)\,,\quad K_1'(\lambda,z)=\beta K_2(\lambda,z)\,, \tag{10}$$

$$K_1\big|_{z=1}=K_1\big|_{z=-1}=0\,. \tag{11}$$

The solution of (10) with the boundary conditions (11) has the form

$$K_1(\lambda,z)=\cos(\lambda_n z)\,,\quad K_2(\lambda,z)=\sin(\lambda_n z)\,, \tag{12}$$

where

$$\lambda_n=\frac{\pi}{2}+\pi n,\quad n=0,1,2,\ldots \tag{13}$$

Hence, the inverse integral transform for (5)-(6) has the form

$$U(y,z)=\sum_{n=0}^{\infty}\bar{u}(\lambda_n,y)\cdot\cos(\lambda_n z)\,, \tag{14}$$

$$b_x(y,z)=\sum_{n=0}^{\infty}\bar{b}(\lambda_n,y)\cdot\sin(\lambda_n z)\,. \tag{15}$$

Then system (7)-(8), describing the problem, takes the form:

$$\frac{d^2\bar{u}}{dy^2}-\lambda_n^2\bar{u}+\frac{2}{\lambda_n}(-1)^n+Ha\lambda_n\bar{b}=0\,, \tag{16}$$

$$\frac{d^2\bar{b}}{dy^2}-\lambda_n^2\bar{b}-Ha\lambda_n\bar{u}=0 \tag{17}$$

with the following boundary conditions, obtained from (4) by using the integral transform (5):

$$y=\pm d:\quad \bar{u}\pm\alpha\frac{d\bar{u}}{dy}=0\,,\quad \bar{b}=0\,. \tag{18}$$

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The following ordinary differential equations for the unknown functions $\bar{u}(\lambda_n, y)$ and $\bar{b}(\lambda_n, y)$ can be obtained from (16)-(17):

$$\bar{u} = \frac{1}{\lambda_n Ha}\left(\bar{b}'' - \lambda_n^2 \bar{b}\right), \quad (19)$$

$$\bar{b}^{(4)} - 2\lambda_n^2 \bar{b}'' + \lambda_n^2\left(\lambda_n^2 + Ha^2\right)\bar{b} = 2(-1)^{n+1} Ha \; . \quad (20)$$

The characteristic equation of the corresponding homogeneous equation of (20) is

$$k^4 - 2\lambda_n^2 k^2 + \lambda_n^2\left(\lambda_n^2 + Ha^2\right) = 0 \,, \quad (21)$$

with the roots

$$k_{1,3} = \pm\sqrt{\lambda_n^2 + i\cdot\lambda_n Ha} \,, \qquad k_{2,4} = \pm\sqrt{\lambda_n^2 - i\cdot\lambda_n Ha} \,. \quad (22)$$

Taking into account that the function $\bar{b}(\lambda_n, y)$ is even with respect to y, the solution of (20) takes the form

$$\bar{b}(\lambda_n, y) = A\cosh\left(k_1 y\right) + B\cosh\left(k_2 y\right) + \frac{2(-1)^{n+1} Ha}{\lambda_n^2\left(\lambda_n^2 + Ha^2\right)} \,. \quad (23)$$

Then it follows from (19) and (23) that

$$\bar{u}(\lambda_n, y) = i\left(A\cosh\left(k_1 y\right) - B\cosh\left(k_2 y\right)\right) - \frac{2(-1)^{n+1} Ha}{\lambda_n\left(\lambda_n^2 + Ha^2\right)} \,, \quad (24)$$

where the coefficients *A* and *B* are determined from the boundary conditions (18) and are equal to

$$A = \frac{2(-1)^n}{\left(\lambda_n{}^2 + Ha^2\right)\cdot\lambda_n{}^2} \times$$

$$\times \frac{\left(Ha + i\,\lambda_n\right)\cosh\left(k_2 d\right) + \alpha\, Ha\, k_2 \sinh\left(k_2 d\right)}{\Delta} \,, \quad (25)$$

$$B = \frac{2(-1)^n}{\left(\lambda_n{}^2 + Ha^2\right)\cdot\lambda_n{}^2} \times$$

$$\times \frac{\left(Ha - i\,\lambda_n\right)\cosh\left(k_1 d\right) + \alpha\, Ha\, k_1 \sinh\left(k_1 d\right)}{\Delta} \,. \quad (26)$$

Applying the inverse integral transforms (14)-(15) to the (23)-(24), the solution of the problem (1)-(4) has the form:

$$U = \sum_{n=0}^{\infty} \frac{2(-1)^n i}{\left(\lambda_n^2 + Ha^2\right)\lambda_n^2}\left(\frac{\tilde{A}\cosh\left(k_1 y\right) - \tilde{B}\cosh\left(k_2 y\right)}{\Delta} - i\lambda_n\right)\cos(\lambda_n z) \quad (27)$$

$$b_x = \sum_{n=0}^{\infty} \frac{2(-1)^n}{\left(\lambda_n^2 + Ha^2\right)\lambda_n^2}\left(\frac{\tilde{A}\cosh\left(k_1 y\right) + \tilde{B}\cosh\left(k_2 y\right)}{\Delta} - Ha\right)\sin(\lambda_n z) \quad (28)$$

where

$$\tilde{A} = \left(Ha + i\cdot\lambda_n\right)\cosh\left(k_2 d\right) + \alpha\cdot Ha\cdot k_2 \sinh\left(k_2 d\right) \,, \quad (29)$$

$$\tilde{B} = \left(Ha - i\cdot\lambda_n\right)\cosh\left(k_1 d\right) + \alpha\cdot Ha\cdot k_1 \sinh\left(k_1 d\right) \,, \quad (30)$$

$$\Delta = 2\cosh\left(k_1 d\right)\cosh\left(k_2 d\right) + \alpha\cdot k_2 \cosh\left(k_1 d\right)\sinh\left(k_2 d\right) +$$

$$+ \alpha\cdot k_1 \cosh\left(k_2 d\right)\sinh\left(k_1 d\right) \,, \quad (31)$$

$$k_1 = \sqrt{\lambda_n^2 + i\cdot\lambda_n Ha} \,, \qquad k_2 = \sqrt{\lambda_n^2 - i\cdot\lambda_n Ha} \,.$$

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